A family of mixed double-Goldberg 6R linkages

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A complete family of double-Goldberg 6R linkages is reported in this article by combining a subtractive Goldberg 5R linkage and a Goldberg 5R linkage through the common link-pair or common Bennett-linkage method. A number of distinct types of overconstrained linkages are built, namely the mixed double-Goldberg 6R linkages. They all have one degree of freedom and their closure equations are derived in detail. One of them degenerates into a Goldberg 5R linkage. From the construction process and geometry conditions, the corresponding relationship between the newly found 6R linkages and the double-Goldberg 6R linkages, constructed from two Goldberg 5R linkages or two subtractive Goldberg 5R linkages, has been established.

Keywords: Goldberg 5R linkage; double-Goldberg 6R linkage; overconstrained linkage; common link-pair method; common Bennett-linkage method

1. Introduction

Various research has been devoted to the invention of single degree-of-freedom overconstrained linkages by combining two or more existing overconstrained linkages. Among them, the Bennett linkage has been a popular construction element since it was proposed in 1903 (Bennett 1903, 1914). Myard (1931) was the first to form 5R and 6R overconstrained linkages with two Bennett linkages. Later, Goldberg (1943) built a family of 5R and 6R linkages with two or three Bennett linkages.

For the 5R linkages, Baker (1979) re-examined both Myard linkages and the Goldberg 5R linkage. He pointed out that the former can be considered as a special case of the latter. A generalized Goldberg 5R linkage, initially introduced by Goldberg, was derived by Wohlhart (1991a) in detail. Recently, Lee (2002) gave an investigation into the kinematics of the generalized Goldberg 5R linkage. Song & Chen (2011) proposed a subtractive Goldberg 5R linkage and investigated its kinematic properties.

For the 6R linkages, several linkages were found by different researchers using a combination construction method. Waldron (1968) merged two Bennett linkages on a common joint and constrained the relative positioning of the links from these two Bennett linkages to build one of his hybrid 6R linkages. Yu & Baker (1981)
Table 1. Notation.

<table>
<thead>
<tr>
<th>notation</th>
<th>description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{(i-1)i}$</td>
<td>length of link $(i - 1)i$, also called link length</td>
</tr>
<tr>
<td>$a_{(i-1)i}$</td>
<td>twist of link $(i - 1)i$, also called link twist</td>
</tr>
<tr>
<td>$R_i$</td>
<td>offset of joint $i$</td>
</tr>
<tr>
<td>$\theta_i$</td>
<td>revolute variable of joint $i$</td>
</tr>
<tr>
<td>$a, b, c, d$</td>
<td>the parameters for link lengths</td>
</tr>
<tr>
<td>$\alpha, \beta, \gamma, \delta$</td>
<td>the parameters for link twists</td>
</tr>
<tr>
<td>$a/\alpha, b/\beta, c/\gamma, d/\delta$</td>
<td>represent the length and twist of a link. For instance, $a/\alpha$ is a link with length $a$ and twist $\alpha$</td>
</tr>
<tr>
<td>G, G1, G2, G3</td>
<td>the Goldberg 5R linkages used in the construction process</td>
</tr>
<tr>
<td>S, S1, S2, S3</td>
<td>the subtractive Goldberg 5R linkages used in the construction process</td>
</tr>
</tbody>
</table>

reported a syncopation of Waldron’s hybrid 6R linkage when they degenerated the 6R linkage into a Goldberg 5R linkage. Later, Baker (1993) further derived two variants of Goldberg 6R linkages. Wohlhart (1991a) found a new 6R linkage through the combination of two Goldberg 5R linkages merged in a ‘face-to-face’ configuration and then removing the commonly shared links and joint. This 6R linkage was further combined with a Bennett linkage to form another 6R linkage with line symmetry (Wohlhart 1991b), in which the combination method is called isomerization. Chen & You (2007) also found a new 6R linkage through the combination of two Goldberg 5R linkages in a ‘back-to-back’ configuration. Similar to Wohlhart’s double-Goldberg 6R linkage, the 6R linkage can also be combined with a Bennett linkage to form a line-symmetric 6R linkage, which is certainly a special case of Bricard line-symmetric linkage (Bricard 1927). Recently, Baker (2009) used Wohlhart’s isomerization method to construct three variants of Bricard line-symmetric 6R linkage with special geometric conditions of the same Bennett ratio on a pair of adjacent links. The same method was applied to find isomeric variants of Dietmaier’s 6R linkage (Baker 2010). Song & Chen (2011) used two subtractive Goldberg 5R linkages as the construction element to form 6R linkage through the common Bennett-linkage (CBL) method.

In this article, a family of mixed double-Goldberg 6R linkages are constructed by connecting one subtractive Goldberg 5R linkage and one Goldberg 5R linkage through either the common link-pair (CLP) or CBL method. The layout of this article is as follows. Section 2 introduces the Goldberg 5R linkage and the subtractive Goldberg 5R linkage. Two construction methods and all possible resultant 6R linkages are presented in §3. In §4, the six distinct types of mixed double-Goldberg 6R linkages are analysed individually to obtain the closure equations. The properties and extensions of this new linkage family are discussed in §5, which concludes the paper. The detailed notations are given in table 1.

2. The Goldberg 5R linkage and the subtractive Goldberg 5R linkage

Initially, Goldberg (1943) built the Goldberg 5R linkage by combining two Bennett linkages in such a way that a link, $a/\alpha$, common to both linkages was removed and a pair of adjacent links, $b/\beta$ and $c/\gamma$, were rigidly attached to
each other, as shown in figure 1. Its geometric conditions and closure equations (Baker 1979) are

\[
\begin{align*}
    a_{12} &= a_{34}, \\
    a_{23} &= a_{45} + a_{51}, \\
    \alpha_{12} &= \alpha_{34}, \\
    \alpha_{23} &= \alpha_{45} + \alpha_{51}, \\
    \sin \alpha_{12} &= \frac{\sin \alpha_{45}}{a_{12}} = \frac{\sin \alpha_{51}}{a_{45}} = \frac{\sin \alpha_{51}}{a_{51}}, \\
    R_i &= 0 \ (i = 1, 2, \ldots, 5)
\end{align*}
\]

(2.1)

and

\[
\begin{align*}
    \tan \frac{\theta_2}{2} &= \frac{\sin((\alpha_{51} + \alpha_{12})/2)}{\tan(\theta_1/2) \sin((\alpha_{51} - \alpha_{12})/2)}, \\
    \tan \frac{\theta_3}{2} &= \frac{\tan(\theta_1/2) \sin((\alpha_{45} + \alpha_{12})/2)}{\sin((\alpha_{45} - \alpha_{12})/2)}, \\
    \theta_1 + \theta_4 &= \pi \quad \text{and} \quad \theta_2 + \theta_3 + \theta_5 = \pi.
\end{align*}
\]

(2.2)

Similarly, with the same two Bennett linkages used in the construction of the Goldberg 5R linkage, a subtractive Goldberg 5R linkage is obtained when links b/β and c/γ are inversely posed, as shown in figure 2 (Song & Chen 2011). The corresponding geometric conditions and closure equations are

\[
\begin{align*}
    a_{12} &= a_{34}, \\
    a_{23} &= a_{45} - a_{51}, \\
    \alpha_{12} &= \alpha_{34}, \\
    \alpha_{23} &= \alpha_{45} - \alpha_{51}, \\
    \sin \alpha_{12} &= \frac{\sin \alpha_{45}}{a_{12}} = \frac{\sin \alpha_{51}}{a_{45}} = \frac{\sin \alpha_{51}}{a_{51}}, \\
    R_i &= 0 \ (i = 1, 2, \ldots, 5)
\end{align*}
\]

(2.3)

and

\[
\begin{align*}
    \tan \frac{\theta_2}{2} &= \frac{\sin((\alpha_{51} + \alpha_{12})/2)}{\tan(\theta_1/2) \sin((\alpha_{51} - \alpha_{12})/2)}, \\
    \tan \frac{\theta_3}{2} &= \frac{\tan(\theta_1/2) \sin((\alpha_{45} + \alpha_{12})/2)}{\sin((\alpha_{45} - \alpha_{12})/2)}, \\
    \theta_1 + \theta_4 &= 2\pi \quad \text{and} \quad \theta_2 + \theta_3 + \theta_5 = 2\pi.
\end{align*}
\]

(2.4)
3. The construction methods

There are two different construction methods reported by Wohlhart (1991a) and Song & Chen (2011) to combine two Goldberg 5R linkages into a double-Goldberg 6R linkage. Both methods request that two 5R linkages contain an identical link-pair. Here, a link-pair is referred to two links connected by a revolute joint. The geometric parameters of two links in the identical link-pair are set as \( a/\alpha \) and \( c/\gamma \), whereas the other links in two 5R linkages are \( b/\beta \) and \( d/\delta \), respectively (figure 3). These four links share the same Bennett ratio according to equations (2.1) and (2.3), i.e.

\[
\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c} = \frac{\sin \delta}{d}. \tag{3.1}
\]

Note that for simplicity, solid dots are used to present the revolute joints in figures 3 and 4 and tables 2 and 3. As shown in figure 3a, Wohlhart (1991a) merged two Goldberg 5R linkages together and removed the commonly shared identical link-pair \( a/\alpha \sim c/\gamma \) to obtain his double-Goldberg linkage. So here we call such method the CLP method. In figure 3b, two identical link-pairs from two Goldberg 5R linkages are constructed into a Bennett linkage shown by dashed lines. Removal of this commonly shared Bennett linkage results in a double-Goldberg 6R linkage (Song & Chen 2011), and so the latter method is called the CBL method.

Here, two different 5R linkages will be used as the construction elements of the new linkage family, a subtractive Goldberg 5R linkage and a Goldberg 5R linkage, namely linkage S and linkage G in the forthcoming derivation. In order to form the CLP or CBL, the identical link-pair in linkages S and G are comprised links \( a/\alpha \) and \( c/\gamma \). Considering that link pairs \( a/\alpha \sim c/\gamma \) can be placed at different positions of two 5R linkages, there are three possible layouts of linkage G, linkages G1, G2 and G3, whereas three possible layouts of linkage S, linkages S1, S2 and S3, as shown in figure 4, in which the identical link-pairs are shown in grey.

For the 6R linkages constructed by combining one linkage S and one linkage G through the CLP or CBL method, there are a total of 18 (i.e. \( 3 \times 3 \times 2 \)) possible combinations. However, after careful analysis and examination, we find that some of the 18 combinations form identical linkages. Thus, only six distinct types of linkages can be constructed with proposed linkages and methods, which are listed in table 2.
Figure 3. Construction of Wohlhart's double-Goldberg 6R linkage using (a) CLP method; (b) CBL method.

Figure 4. Schematic of the possible linkages S and G with identical link-pair. (a) Linkages S1, S2 and S3; (b) linkages G1, G2 and G3, in which the identical link-pair $a/\alpha \sim c/\gamma$ is marked in grey lines.
Table 2. All possible constructions of the mixed double-Goldberg 6R linkages.

<table>
<thead>
<tr>
<th>linkage type</th>
<th>S</th>
<th>G</th>
<th>connection method</th>
<th>schematic example</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>S3</td>
<td>G3</td>
<td>CLP and CBL</td>
<td><img src="image" alt="schematic example" /></td>
</tr>
<tr>
<td>II</td>
<td>S3</td>
<td>G1</td>
<td>CLP and CBL</td>
<td><img src="image" alt="schematic example" /></td>
</tr>
<tr>
<td></td>
<td>S3</td>
<td>G2</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>S1</td>
<td>G3</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>S2</td>
<td>G3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>III</td>
<td>S1</td>
<td>G1</td>
<td>CLP</td>
<td><img src="image" alt="schematic example" /></td>
</tr>
<tr>
<td></td>
<td>S2</td>
<td>G2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>IV</td>
<td>S1</td>
<td>G1</td>
<td>CBL</td>
<td><img src="image" alt="schematic example" /></td>
</tr>
<tr>
<td></td>
<td>S2</td>
<td>G2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>V</td>
<td>S1</td>
<td>G2</td>
<td>CLP</td>
<td><img src="image" alt="schematic example" /></td>
</tr>
<tr>
<td></td>
<td>S2</td>
<td>G1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### 4. Six types of mixed double-Goldberg 6R linkages

As the six distinct linkages listed in Table 2 are built from two different types of 5R linkages, linkages S and G, we name them as a family of mixed double-Goldberg 6R linkages. Here, only the type I linkage is used as an example with detailed derivation of the closure equations, the rest can be obtained similarly. The closure equations, $m_1$ to $m_6$ are introduced to shorten the lengthy equations, where

$$
m_1 = \frac{\sin((\beta + \alpha)/2)}{\sin((\beta - \alpha)/2)}, \quad m_2 = \frac{\sin((\gamma + \alpha)/2)}{\sin((\gamma - \alpha)/2)}, \quad m_3 = \frac{\sin((\delta + \alpha)/2)}{\sin((\delta - \alpha)/2)},
$$

$$
m_4 = \frac{\sin((\delta + \gamma)/2)}{\sin((\delta - \gamma)/2)}, \quad m_5 = \frac{\sin((\delta + \beta)/2)}{\sin((\delta - \beta)/2)}, \quad m_6 = \frac{\sin((\beta + \gamma)/2)}{\sin((\beta - \gamma)/2)}.
$$

(a) Type I mixed double-Goldberg 6R linkage

Linkages S3 and G3 are selected to build the type I linkage. The identical link-pairs $a/\alpha \sim c/\gamma$ are link-pair 51–45 of both linkages S3 and G3. So, the geometry conditions of linkages S3 and G3 are

$$
\begin{align*}
a_{12}^{S3} &= a_{34}^{S3} = b, & a_{23}^{S3} &= a - c, & a_{35}^{S3} &= c, & a_{51}^{S3} &= a, \\
\alpha_{12}^{S3} &= \alpha_{34}^{S3} = \beta, & \alpha_{23}^{S3} &= \alpha - \gamma, & \alpha_{35}^{S3} &= \gamma, & \alpha_{51}^{S3} &= \alpha, \\
a_{12}^{G3} &= a_{34}^{G3} = d, & a_{23}^{G3} &= a + c, & a_{35}^{G3} &= c, & a_{51}^{G3} &= a \\
\alpha_{12}^{G3} &= \alpha_{34}^{G3} = \delta, & \alpha_{23}^{G3} &= \alpha + \gamma, & \alpha_{35}^{G3} &= \gamma, & \alpha_{51}^{G3} &= \alpha,
\end{align*}
$$

and

$$
\alpha_{12}^{G3} = \alpha_{34}^{G3} = \delta, \quad \alpha_{23}^{G3} = \alpha + \gamma, \quad \alpha_{35}^{G3} = \gamma, \quad \alpha_{51}^{G3} = \alpha.
$$
Figure 5. Construction of the type I mixed double-Goldberg 6R linkage.

Using the CLP method, a 6R linkage can be obtained from linkages S3 and G3 (figure 5). The geometry conditions of the resultant 6R linkage are

\[
\begin{align*}
& a_{12} = a - c, \quad a_{23} = a_{61} = b, \quad a_{34} = a_{56} = d, \quad a_{45} = a + c, \\
& \alpha_{12} = \alpha - \gamma, \quad \alpha_{23} = \alpha_{61} = \beta, \quad \alpha_{34} = \alpha_{56} = \delta, \quad \alpha_{45} = \alpha + \gamma, \\
& \frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c} = \frac{\sin \delta}{d}, \\
& R_i = 0 \quad (i = 1, 2, \ldots, 6).
\end{align*}
\]

According to equations (2.2) and (2.4), the closure equations of linkages S3 and G3 can be written as

\[
\begin{align*}
& \tan \frac{\theta_{2}^{S3}}{2} = -\frac{m_2}{\tan(\theta_{1}^{S3}/2)}, \quad \tan \frac{\theta_{3}^{S3}}{2} = \frac{\tan(\theta_{1}^{S3}/2)}{m_6}, \\
& \theta_1^{S3} + \theta_4^{S3} = 2\pi \quad \text{and} \quad \theta_2^{S3} + \theta_3^{S3} + \theta_5^{S3} = 2\pi;
\end{align*}
\]

and

\[
\begin{align*}
& \tan \frac{\theta_{2}^{G3}}{2} = -\frac{m_2}{\tan(\theta_{1}^{G3}/2)}, \quad \tan \frac{\theta_{3}^{G3}}{2} = m_4 \tan \frac{\theta_{1}^{G3}}{2}, \\
& \theta_1^{G3} + \theta_4^{G3} = \pi \quad \text{and} \quad \theta_2^{G3} + \theta_3^{G3} + \theta_5^{G3} = \pi
\end{align*}
\]

respectively. The relationship between revolute variables of the resultant 6R linkage and linkages S3, G3 are

\[
\begin{align*}
& \theta_1 = \theta_{2}^{S3}, \quad \theta_2 = \theta_{3}^{S3}, \quad \theta_3 = \theta_{4}^{S3} - \theta_{4}^{G3} + \pi, \\
& \theta_4 = 2\pi - \theta_{3}^{G3}, \quad \theta_5 = 2\pi - \theta_{2}^{G3}, \quad \theta_6 = \pi + \theta_{1}^{S3} - \theta_{1}^{G3}.
\end{align*}
\]
In addition, the compatibility condition,

\[ \theta_5^{S3} = \theta_5^{G3}, \]  

(4.7)

must hold to build a successful connection. Substituting equations (4.6) and (4.7) into equations (4.4) and (4.5) gets the closure equations of the type I linkage,

\[
\begin{align*}
\frac{\tan \theta_2}{2} &= -\frac{m_2}{m_6 \tan(\theta_1/2)}, \\
\theta_3 &= \pi + 2 \tan^{-1} Q - 2 \tan^{-1}\left(\frac{m_2}{\tan(\theta_1/2)}\right), \quad \frac{\tan \theta_4}{2} = -\frac{m_2}{Q} \\
\frac{\tan \theta_5}{2} &= m_4 \tan \frac{\theta_1}{2}, \quad \theta_6 = 2 \tan^{-1}\left(\frac{m_2}{\tan(\theta_1/2)}\right) - 2 \tan^{-1} Q,
\end{align*}
\]

(4.8)

in which

\[
Q = \begin{cases} 
\frac{1}{2H} \left[ -\left(\frac{1}{m_2} + m_4\right) + \sqrt{\left(\frac{1}{m_2} + m_4\right)^2 + 4H^2 \frac{m_2}{m_4}} \right] \\
(\theta_1 \in [2n\pi, \pi + 2n\pi]) 
\end{cases}
\]

(4.9)

\[
H = \frac{m_2 - m_6 \tan^2(\theta_1/2)}{(m_2 + m_6) \tan(\theta_1/2)}.
\]

The input–output curves of the type I linkage are plotted in figure 6.

It should be pointed out that using the CBL method with linkages S3 and G3 in the same configuration, the same 6R linkage can be formed but in a different configuration. In figure 6, the hollow dots represent the configuration of the 6R linkage from the CLP method at \( \theta_1 = \theta_1^{CLP} \), and the solid dots represent the configuration of the 6R linkage from the CBL method at \( \theta_1 = \theta_1^{CBL} \), with \( \theta_1^{CLP} = -\theta_1^{CBL} \) for all configurations.

(b) Type II mixed double-Goldberg 6R linkage

Linkages S2 and G3 are selected to build the type II linkage. As shown in figure 7, the identical link-pairs \( a/\alpha \sim c/\gamma \) are link-pair 34–45 of linkage S2 and
link-pair 51–45 of linkage G3. Then the geometry conditions of linkages S2 and G3 are

\[
\begin{align*}
    a_{12}^{S2} &= a_{34}^{S2} = a, & a_{23}^{S2} &= b - c, & a_{45}^{S2} &= c, & a_{51}^{S2} &= b, \\
    a_{12}^{G3} &= a_{34}^{G3} = a, & a_{23}^{G3} &= b - \gamma, & a_{45}^{G3} &= \gamma, & a_{51}^{G3} &= \beta; \\
    a_{12}^{G3} &= a_{34}^{G3} = d, & a_{23}^{G3} &= a + c, & a_{45}^{G3} &= c, & a_{51}^{G3} &= a, \\
    a_{12}^{G3} &= a_{34}^{G3} = \delta, & a_{23}^{G3} &= \alpha + \gamma, & a_{45}^{G3} &= \gamma, & a_{51}^{G3} &= \alpha.
\end{align*}
\]

(4.10)
Using the CLP method with linkages S2 and G3, a 6R linkage can be formed with the geometry conditions as

\[
\begin{aligned}
    a_{12} &= a, & a_{23} &= b - c, & a_{34} &= a_{56} = d, & a_{45} &= a + c, & a_{61} &= b, \\
    \alpha_{12} &= \alpha, & \alpha_{23} &= \beta - \gamma, & \alpha_{34} &= \alpha_{56} = \delta, & \alpha_{45} &= \alpha + \gamma, & \alpha_{61} &= \beta,
\end{aligned}
\]

\[
\begin{align*}
    \sin \alpha &= \frac{\sin \beta}{b} = \frac{\sin \gamma}{c} = \frac{\sin \delta}{d}
\end{align*}
\]

and

\[
R_i = 0 \quad (i = 1, 2, \ldots, 6).
\]

The closure equations of the type II linkage can be derived as follows and its input–output curves are plotted in figure 8.

\[
\begin{align*}
\frac{\tan \theta_2}{2} &= \frac{m_1}{\tan(\theta_1/2)}, & \frac{\tan \theta_3}{2} &= \frac{P \cdot \tan(\theta_1/2) - m_2}{\tan(\theta_1/2) + P \cdot m_2}, \\
\frac{\tan \theta_4}{2} &= -\frac{m_3}{P}, & \frac{\tan \theta_5}{2} &= -P \cdot m_4
\end{align*}
\]

and

\[
\frac{\tan \theta_6}{2} = \frac{\tan^2(\theta_1/2) + m_1 m_2 - P \cdot (m_1 - m_2) \tan(\theta_1/2)}{(m_1 - m_2) \tan(\theta_1/2) + P \cdot (m_1 m_2 + \tan^2(\theta_1/2))},
\]

in which

\[
P = \frac{1}{2 m_4 \tan(\theta_1/2)} \left[ (m_3 m_4 - 1) \pm \sqrt{(m_3 m_4 - 1)^2 - 4 m_3 m_4 \tan^2(\theta_1/2)} \right].
\]

As shown in table 2, the same linkage can be obtained when using the CBL method. On the input–output curves in figure 8, the linkage obtained from the CLP method (hollow dots) is at the configuration \( \theta_1 = \theta_1^{\text{CLP}} \), whereas the linkage
obtained from the CBL method (solid dots) is at the configuration \( \theta_1 = \theta_1^{\text{CBL}} \). Then \( \theta_1^{\text{CBL}} = \theta_1^{\text{CLP}} + 2\pi \) is always hold for the same configurations of linkages S2 and G3. It is interesting to note that \( \theta_1 \) needs to rotate two full circles, or variate in the range of \([-2\pi, 2\pi)\), to make \( \theta_3, \theta_4, \theta_5 \) and \( \theta_6 \) have one full circle of motion, while \( \theta_2 \) follows with two circles. As given in table 2, the combination between S3 and G1, S3 and G2 or S1 and G3 result in the same 6R linkage.

(c) Types III and IV mixed double-Goldberg 6R linkages

Linkages S2 and G2 are selected to build both types III and IV linkages. Link-pair 34–45 of both linkages are the identical link-pairs. The geometry conditions of linkages S2 and G2 are

\[
\begin{align*}
\alpha_{S2}^{12} &= \alpha_{S2}^{34} = \alpha, & & \alpha_{S2}^{23} &= b - c, & & \alpha_{S2}^{45} &= c, & & \alpha_{S2}^{51} &= b, \\
\alpha_{G2}^{12} &= \alpha_{G2}^{34} = \alpha, & & \alpha_{G2}^{23} &= \beta - \gamma, & & \alpha_{G2}^{45} &= \gamma, & & \alpha_{G2}^{51} &= \beta; \\
\alpha_{S2}^{12} &= \alpha_{S2}^{34} = \alpha, & & \alpha_{S2}^{23} &= c + d, & & \alpha_{S2}^{45} &= c, & & \alpha_{S2}^{51} &= d, \\
\alpha_{G2}^{12} &= \alpha_{G2}^{34} = \alpha, & & \alpha_{G2}^{23} &= \gamma + \delta, & & \alpha_{G2}^{45} &= \gamma, & & \alpha_{G2}^{51} &= \delta.
\end{align*}
\]

(4.13)

Using the CLP method, type III linkage can be obtained (figure 9). Then its geometry conditions are

\[
\begin{align*}
\alpha_{12} &= \alpha_{45} = \alpha, & & \alpha_{23} &= b - c, & & \alpha_{34} &= c + d, & & \alpha_{56} &= d, & & \alpha_{61} &= b, \\
\alpha_{12} &= \alpha_{45} = \alpha, & & \alpha_{23} &= \beta - \gamma, & & \alpha_{34} &= \gamma + \delta, & & \alpha_{56} &= \delta, & & \alpha_{61} &= \beta, \\
\frac{\sin \alpha}{a} &= \frac{\sin \beta}{b} = \frac{\sin \gamma}{c} = \frac{\sin \delta}{d}.
\end{align*}
\]

(4.14)

and

\[
R_i = 0 \quad (i = 1, 2, \ldots, 6).
\]

When using the CBL method with linkages S2 and G2, a type IV linkage is obtained (figure 10). Thus, its geometry conditions are

\[
\begin{align*}
\alpha_{12} &= \alpha_{45} = \alpha, & & \alpha_{23} &= b - c, & & \alpha_{34} &= d, & & \alpha_{56} &= c + d, & & \alpha_{61} &= b, \\
\alpha_{12} &= \alpha_{45} = \alpha, & & \alpha_{23} &= \beta - \gamma, & & \alpha_{34} &= \delta, & & \alpha_{56} &= \gamma + \delta, & & \alpha_{61} &= \beta, & & \frac{\sin \alpha}{a} &= \frac{\sin \beta}{b} = \frac{\sin \gamma}{c} = \frac{\sin \delta}{d}.
\end{align*}
\]

(4.15)

and

\[
R_i = 0 \quad (i = 1, 2, \ldots, 6).
\]
Mixed double-Goldberg 6R linkages

Figure 10. Construction of the type IV mixed double-Goldberg 6R linkage.

The closure equations of type III and IV linkages can be derived as

\[
\begin{align*}
\tan \frac{\theta_2}{2} &= \frac{m_1}{\tan(\theta_1/2)}, \quad \theta_3 = 0, \quad \tan \frac{\theta_4}{2} = -m_3 \tan \frac{\theta_1}{2}, \\
\theta_5 &= \pi - \theta_1 \quad \text{and} \quad \tan \frac{\theta_6}{2} = \frac{(1 - m_1 m_3) \tan(\theta_1/2)}{m_1 + m_3 \tan^2(\theta_1/2)}
\end{align*}
\]

and

\[
\begin{align*}
\tan \frac{\theta_2}{2} &= \frac{m_1}{\tan(\theta_1/2)}, \quad \tan \frac{\theta_3}{2} = \frac{m_2 + m_3 \tan^2(\theta_1/2)}{(m_2 m_3 - 1) \tan(\theta_1/2)}, \quad \theta_4 = \pi - \theta_1, \\
\tan \frac{\theta_5}{2} &= m_3 \tan \frac{\theta_1}{2} \quad \text{and} \quad \tan \frac{\theta_6}{2} = \frac{m_1 m_2 + \tan^2(\theta_1/2)}{(m_1 - m_2) \tan(\theta_1/2)}
\end{align*}
\]

respectively. Their input–output curves are plotted in figures 11 and 12. For the type III linkage, it is obvious that \( \theta_3 \) is constrained to zero during the full circle of motion. The instantaneous mobility of joint 3 is locked to zero, i.e. the link-pair 23–34 could be viewed as a composite link 24 of \( b + d/\beta + \delta \). Therefore, the type III linkage is equivalent to a Goldberg 5R linkage. As demonstrated in table 2, the connection of linkages S1 and G1 gives the same 6R linkages.

(d) Types V and VI mixed double-Goldberg 6R linkages

Linkages S2 and G1 are selected to build both types V and VI linkages. Link-pair 34–45 of linkage S2 and link-pair 51–12 of linkage G1 are the identical
The geometry conditions of linkages S2 and G1 are

\[
\begin{align*}
    a^{S2}_{12} &= a^{S2}_{34} = a, & a^{S2}_{23} &= b - c, & a^{S2}_{45} &= c, & a^{S2}_{51} &= b, \\
    a^{G1}_{12} &= a^{G1}_{34} = a, & a^{G1}_{23} &= b - d, & a^{G1}_{45} &= d, & a^{G1}_{51} &= a, \\
    \alpha^{S2}_{12} &= \alpha^{S2}_{34} = \alpha, & \alpha^{S2}_{23} &= \beta - \gamma, & \alpha^{S2}_{45} &= \gamma, & \alpha^{S2}_{51} &= \beta; \\
    \alpha^{G1}_{12} &= \alpha^{G1}_{34} = \gamma, & \alpha^{G1}_{23} &= \alpha + \delta, & \alpha^{G1}_{45} &= \delta, & \alpha^{G1}_{51} &= \alpha.
\end{align*}
\] (4.18)

Figure 11. The input–output curves of the type III mixed double-Goldberg 6R linkage.

Figure 12. The input–output curves of the type IV mixed double-Goldberg 6R linkage.
Using the CLP method a type V linkage is obtained, as shown in figure 13. Its geometry conditions are

$$\begin{align*}
    a_{12} &= a, & a_{23} &= b - c, & a_{34} &= d, & a_{45} &= c, & a_{56} &= a + d, & a_{61} &= b, \\
    \alpha_{12} &= \alpha, & \alpha_{23} &= \beta - \gamma, & \alpha_{34} &= \delta, & \alpha_{45} &= \gamma, & \alpha_{56} &= \alpha + \delta, & \alpha_{61} &= \beta, \\
    \sin \alpha &= \frac{\sin \beta}{a} = \frac{\sin \gamma}{b} = \frac{\sin \delta}{c} = \frac{\sin \delta}{d} \\
    \text{and} & \quad R_i = 0 \quad (i = 1, 2, \ldots, 6).
\end{align*}$$

(4.19)

Similar to types III and IV, a type VI linkage can be obtained when using the CBL method with linkages S2 and G1 (figure 14). The geometry conditions of the type VI linkage are

$$\begin{align*}
    a_{12} &= a, & a_{23} &= b - c, & a_{34} &= a + d, & a_{45} &= c, & a_{56} &= d, & a_{61} &= b, \\
    \alpha_{12} &= \alpha, & \alpha_{23} &= \beta - \gamma, & \alpha_{34} &= \alpha + \delta, & \alpha_{45} &= \gamma, & \alpha_{56} &= \delta, & \alpha_{61} &= \beta, \\
    \sin \alpha &= \frac{\sin \beta}{a} = \frac{\sin \gamma}{b} = \frac{\sin \delta}{c} = \frac{\sin \delta}{d} \\
    \text{and} & \quad R_i = 0 \quad (i = 1, 2, \ldots, 6).
\end{align*}$$

(4.20)

The closure equations of the types V and VI linkages can be derived as

$$\begin{align*}
    \tan \frac{\theta_2}{2} &= \frac{m_1}{\tan(\theta_1/2)}, & \tan \frac{\theta_3}{2} &= \frac{1}{m_4 \tan(\theta_1/2)}, \\
    \theta_4 &= \pi - \theta_1, \theta_5 = \pi - \theta_3 \quad \text{and} \quad \tan \frac{\theta_6}{2} = -\frac{m_1}{\tan(\theta_1/2)}
\end{align*}$$

(4.21)
Figure 14. Construction of the type VI mixed double-Goldberg 6R linkage.

Figure 15. The input–output curves of the type V mixed double-Goldberg 6R linkage.

\[
\tan \frac{\theta_2}{2} = \frac{m_1}{\tan(\theta_1/2)}; \quad \tan \frac{\theta_3}{2} = \frac{\tan(\theta_1/2)}{m_2},
\]

\[
\tan \frac{\theta_4}{2} = m_4 \tan \frac{\theta_1}{2}; \quad \theta_5 = \pi - \theta_1 \quad \text{and}
\]

\[
\tan \frac{\theta_6}{2} = \frac{(m_4 \tan^2(\theta_1/2) + m_1 m_2 m_4 + m_1 - m_2) \tan(\theta_1/2)}{(m_1 m_4 - m_2 m_4 - 1) \tan^2(\theta_1/2) - m_1 m_2};
\]

respectively. Their input–output curves are plotted in figures 15 and 16.
From the geometry conditions, the type V linkage is in fact a Goldberg 6R linkage constructed from one Bennett linkage, with links \( a/\alpha, b/\beta \) and another Bennett linkage with links \( c/\gamma, d/\delta \) connected side by side. It can also be considered as a special case of Waldron’s hybrid 6R linkage (Waldron 1968). The type VI linkage is related to the L-shape Goldberg 6R linkage (Goldberg 1943) where link 23 in the resultant 6R linkage is negative in length (Chen & You 2005, 2008). Moreover, both 6R linkages can also be constructed with linkages S1 and G2, as listed in table 2.

5. Conclusion and discussions

A new family of mixed double-Goldberg 6R linkages have been built by the combination of a subtractive Goldberg 5R linkage and a Goldberg 5R linkage through the CLP or CBL method. All of them have a single degree of freedom. The physical model of the type V linkage is shown in figure 17. From the
Table 3. The complete families of double-Goldberg $6R$ linkages.

<table>
<thead>
<tr>
<th>Type</th>
<th>Mixed Double-Goldberg linkage from</th>
<th>Double-Goldberg linkage from two</th>
<th>Subtractive Double-Goldberg linkage from two S linkages</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td><img src="image1" alt="Diagram" /></td>
<td><img src="image2" alt="Diagram" /></td>
<td><img src="image3" alt="Diagram" /></td>
</tr>
<tr>
<td>II</td>
<td><img src="image4" alt="Diagram" /></td>
<td><img src="image5" alt="Diagram" /></td>
<td></td>
</tr>
<tr>
<td>III</td>
<td><img src="image6" alt="Diagram" /></td>
<td><img src="image7" alt="Diagram" /></td>
<td></td>
</tr>
<tr>
<td>IV</td>
<td><img src="image8" alt="Diagram" /></td>
<td><img src="image9" alt="Diagram" /></td>
<td></td>
</tr>
<tr>
<td>V</td>
<td><img src="image10" alt="Diagram" /></td>
<td><img src="image11" alt="Diagram" /></td>
<td></td>
</tr>
</tbody>
</table>

(Continued.)
Table 3. (Continued.)

<table>
<thead>
<tr>
<th>Type</th>
<th>The Mixed Double-Goldberg Linkage from Linkages S and G</th>
<th>The Double-Goldberg Linkage from Two G Linkages</th>
<th>The Subtractive Double-Goldberg Linkage from Two S Linkages</th>
</tr>
</thead>
<tbody>
<tr>
<td>VI</td>
<td><img src="image" alt="Diagram of the L-shape Goldberg 6R Linkage" /></td>
<td><img src="image" alt="Diagram of the L-shape Goldberg 6R Linkage" /></td>
<td><img src="image" alt="Diagram of the L-shape Goldberg 6R Linkage" /></td>
</tr>
</tbody>
</table>

In this article, we used two different types of Goldberg 5R linkages as construction elements. Alternatively, two Goldberg 5R linkages or two subtractive Goldberg 5R linkages can be used to construct 6R linkages through the CLP or CBL method. The resultant linkages are the double-Goldberg 6R linkages reported previously and the corresponding subtractive cases, respectively, as listed in Table 3. Owing to the identical Bennett ratio in links $a/\alpha$, $b/\beta$ and $d/\delta$, type I of the double-Goldberg 6R linkages from two linkages G, i.e. Wohlhart’s double-Goldberg linkage, can be transformed into the line-symmetric Bricard linkage through isomerization; so does the type I 6R linkage from two linkages S. Thus, such linkages with implicit symmetric characteristics exhibit the similar bifurcation behaviours as the special line-symmetric Bricard linkage without offset. However, the type I mixed double-Goldberg 6R linkage has no such problem as it generally has no explicit or implicit symmetric properties.

All linkages in the family of mixed double-Goldberg 6R linkages are built from four basic links $a/\alpha$, $b/\beta$, $c/\gamma$ and $d/\delta$. After comparing the geometry conditions of each linkage type, an extra link $c/\gamma$ is identified, which plays a role different to that of the other three links ($a/\alpha$, $b/\beta$ and $d/\delta$). In type V and VI linkages, link $c/\gamma$ is one of the six individual links in the linkage, while for the rest, link $c/\gamma$ does not directly exist in the linkage but hides in links, such as $a + c/\alpha + \gamma$, $b - c/\beta - \gamma$, etc. For example, in the type I linkage, when $c/\gamma$ is shrunk to zero, links 12 and 45 will be link $a/\alpha$ at the same time. The type I linkage then becomes a Mavroidis and Roth 6R linkage with zero offsets; similar observations can be found in other linkage types. When link $c/\gamma$ is changed to a negative length, linkages S and G are swapped with each other. The resultant 6R linkages are still the mixed double-Goldberg 6R linkages.

By now, it can be concluded that all possible 6R overconstrained linkages constructed from two Goldberg 5R linkages of the same kind, or two different kinds, through the CLP or the CBL method, have been found and listed in this
article. The relationships among all of the Bennett-based 6R linkages that are involved have been disclosed. This work paves the road for the application of these 6R linkages in the design of reconfigurable mechanisms (Zhang & Dai 2009).

References


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