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A family of mixed double-Goldberg 6*R* linkages

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A complete family of double-Goldberg 6*R* linkages is reported in this article by combining a subtractive Goldberg 5*R* linkage and a Goldberg 5*R* linkage through the common link-pair or common Bennett-linkage method. A number of distinct types of overconstrained linkages are built, namely the mixed double-Goldberg 6*R* linkages. They all have one degree of freedom and their closure equations are derived in detail. One of them degenerates into a Goldberg 5*R* linkage. From the construction process and geometry conditions, the corresponding relationship between the newly found 6*R* linkages and the double-Goldberg 6*R* linkages, constructed from two Goldberg 5*R* linkages or two subtractive Goldberg 5*R* linkages, has been established.

Keywords: Goldberg 5*R* linkage; double-Goldberg 6*R* linkage; overconstrained linkage; common link-pair method; common Bennett-linkage method

1. Introduction

Various research has been devoted to the invention of single degree-of-freedom overconstrained linkages by combining two or more existing overconstrained linkages. Among them, the Bennett linkage has been a popular construction element since it was proposed in 1903 (Bennett 1903, 1914). Myard (1931) was the first to form 5*R* and 6*R* overconstrained linkages with two Bennett linkages. Later, Goldberg (1943) built a family of 5*R* and 6*R* linkages with two or three Bennett linkages.

For the 5*R* linkages, Baker (1979) re-examined both Myard linkages and the Goldberg 5*R* linkage. He pointed out that the former can be considered as a special case of the latter. A generalized Goldberg 5*R* linkage, initially introduced by Goldberg, was derived by Wohlhart (1991*a*) in detail. Recently, Lee (2002) gave an investigation into the kinematics of the generalized Goldberg 5*R* linkage. Song & Chen (2011) proposed a subtractive Goldberg 5*R* linkage and investigated its kinematic properties.

For the 6*R* linkages, several linkages were found by different researchers using a combination construction method. Waldron (1968) merged two Bennett linkages on a common joint and constrained the relative positioning of the links from these two Bennett linkages to build one of his hybrid 6*R* linkages. Yu & Baker (1981)

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Table 1. Notation.

$a_{(i-1)i}$	length of link $(i-1)i$, also called link length
$\alpha_{(i-1)i}$	twist of link $(i-1)i$, also called link twist
R_i	offset of joint i
θ_i	revolute variable of joint i
a, b, c, d	the parameters for link lengths
α, β, γ, δ	the parameters for link twists
a/α, b/β, c/γ, d/δ	represent the length and twist of a link. For instance, a/α is a link with length a and twist α
G, G1, G2, G3	the Goldberg $5R$ linkages used in the construction process
S, S1, S2, S3	the subtractive Goldberg $5R$ linkages used in the construction process

reported a syncopation of Waldron's hybrid $6R$ linkage when they degenerated the $6R$ linkage into a Goldberg $5R$ linkage. Later, Baker (1993) further derived two variants of Goldberg $6R$ linkages. Wohlhart (1991*a*) found a new $6R$ linkage through the combination of two Goldberg $5R$ linkages merged in a 'face-to-face' configuration and then removing the commonly shared links and joint. This $6R$ linkage was further combined with a Bennett linkage to form another $6R$ linkage with line symmetry (Wohlhart 1991*b*), in which the combination method is called isomerization. Chen & You (2007) also found a new $6R$ linkage through the combination of two Goldberg $5R$ linkages in a 'back-to-back' configuration. Similar to Wohlhart's double-Goldberg $6R$ linkage, the $6R$ linkage can also be combined with a Bennett linkage to form a line-symmetric $6R$ linkage, which is certainly a special case of Bricard line-symmetric linkage (Bricard 1927). Recently, Baker (2009) used Wohlhart's isomerization method to construct three variants of Bricard line-symmetric $6R$ linkage with special geometric conditions of the same Bennett ratio on a pair of adjacent links. The same method was applied to find isomeric variants of Dietmaier's $6R$ linkage (Baker 2010). Song & Chen (2011) used two subtractive Goldberg $5R$ linkages as the construction element to form $6R$ linkage through the common Bennett-linkage (CBL) method.

In this article, a family of mixed double-Goldberg $6R$ linkages are constructed by connecting one subtractive Goldberg $5R$ linkage and one Goldberg $5R$ linkage through either the common link-pair (CLP) or CBL method. The layout of this article is as follows. Section 2 introduces the Goldberg $5R$ linkage and the subtractive Goldberg $5R$ linkage. Two construction methods and all possible resultant $6R$ linkages are presented in §3. In §4, the six distinct types of mixed double-Goldberg $6R$ linkages are analysed individually to obtain the closure equations. The properties and extensions of this new linkage family are discussed in §5, which concludes the paper. The detailed notations are given in table 1.

2. The Goldberg 5*R* linkage and the subtractive Goldberg 5*R* linkage

Initially, Goldberg (1943) built the Goldberg $5R$ linkage by combining two Bennett linkages in such a way that a link, a/α, common to both linkages was removed and a pair of adjacent links, b/β and c/γ, were rigidly attached to

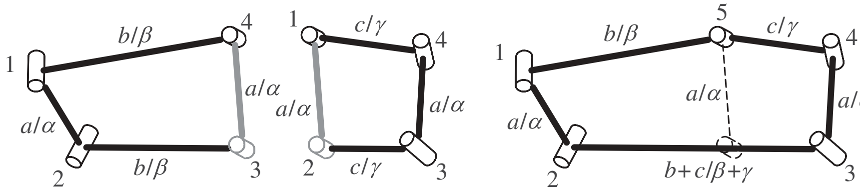


Figure 1. Construction of the Goldberg $5R$ linkage.

each other, as shown in figure 1. Its geometric conditions and closure equations (Baker 1979) are

$$\left.\begin{aligned} &a_{12}=a_{34}, \quad a_{23}=a_{45}+a_{51},\\ &\alpha_{12}=\alpha_{34}, \quad \alpha_{23}=\alpha_{45}+\alpha_{51},\\ &\frac{\sin\alpha_{12}}{a_{12}}=\frac{\sin\alpha_{45}}{a_{45}}=\frac{\sin\alpha_{51}}{a_{51}},\\ &R_i=0\ (i=1,2,\ldots,5) \end{aligned}\right\} \tag{2.1}$$

and

$$\left.\begin{aligned} \tan\frac{\theta_2}{2}&=\frac{\sin((\alpha_{51}+\alpha_{12})/2)}{\tan(\theta_1/2)\sin((\alpha_{51}-\alpha_{12})/2)},\\ \tan\frac{\theta_3}{2}&=\frac{\tan(\theta_1/2)\sin((\alpha_{45}+\alpha_{12})/2)}{\sin((\alpha_{45}-\alpha_{12})/2)},\\ \theta_1+\theta_4=\pi \quad &\text{and} \quad \theta_2+\theta_3+\theta_5=\pi. \end{aligned}\right\} \tag{2.2}$$

Similarly, with the same two Bennett linkages used in the construction of the Goldberg $5R$ linkage, a subtractive Goldberg $5R$ linkage is obtained when links b/β and c/γ are inversely posed, as shown in figure 2 (Song & Chen 2011). The corresponding geometric conditions and closure equations are

$$\left.\begin{aligned} &a_{12}=a_{34}, \quad a_{23}=a_{45}-a_{51},\\ &\alpha_{12}=\alpha_{34}, \quad \alpha_{23}=\alpha_{45}-\alpha_{51},\\ &\frac{\sin\alpha_{12}}{a_{12}}=\frac{\sin\alpha_{45}}{a_{45}}=\frac{\sin\alpha_{51}}{a_{51}},\\ &R_i=0\ (i=1,2,\ldots,5) \end{aligned}\right\} \tag{2.3}$$

and

$$\left.\begin{aligned} \tan\frac{\theta_2}{2}&=\frac{\sin((\alpha_{51}+\alpha_{12})/2)}{\tan(\theta_1/2)\sin((\alpha_{51}-\alpha_{12})/2)},\\ \tan\frac{\theta_3}{2}&=\frac{\tan(\theta_1/2)\sin((\alpha_{45}+\alpha_{12})/2)}{\sin((\alpha_{45}-\alpha_{12})/2)},\\ \theta_1+\theta_4=2\pi \quad &\text{and}\ \theta_2+\theta_3+\theta_5=2\pi. \end{aligned}\right\} \tag{2.4}$$

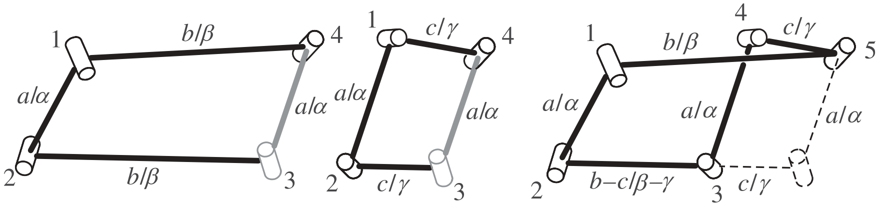


Figure 2. Construction of the subtractive Goldberg $5R$ linkage.

3. The construction methods

There are two different construction methods reported by Wohlhart (1991*a*) and Song & Chen (2011) to combine two Goldberg $5R$ linkages into a double-Goldberg $6R$ linkage. Both methods request that two $5R$ linkages contain an identical link-pair. Here, a link-pair is referred to two links connected by a revolute joint. The geometric parameters of two links in the identical link-pair are set as a/α and c/γ, whereas the other links in two $5R$ linkages are b/β and d/δ, respectively (figure 3). These four links share the same Bennett ratio according to equations (2.1) and (2.3), i.e.

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}=\frac{\sin\delta}{d}. \tag{3.1}$$

Note that for simplicity, solid dots are used to present the revolute joints in figures 3 and 4 and tables 2 and 3. As shown in figure 3*a*, Wohlhart (1991*a*) merged two Goldberg $5R$ linkages together and removed the commonly shared identical link-pair $a/\alpha \sim c/\gamma$ to obtain his double-Goldberg linkage. So here we call such method the *CLP* method. In figure 3*b*, two identical link-pairs from two Goldberg $5R$ linkages are constructed into a Bennett linkage shown by dashed lines. Removal of this commonly shared Bennett linkage results in a double-Goldberg $6R$ linkage (Song & Chen 2011), and so the latter method is called the *CBL* method.

Here, two different $5R$ linkages will be used as the construction elements of the new linkage family, a subtractive Goldberg $5R$ linkage and a Goldberg $5R$ linkage, namely linkage S and linkage G in the forthcoming derivation. In order to form the CLP or CBL, the identical link-pair in linkages S and G are comprised links a/α and c/γ. Considering that link pairs $a/\alpha \sim c/\gamma$ can be placed at different positions of two $5R$ linkages, there are three possible layouts of linkage G, linkages G1, G2 and G3, whereas three possible layouts of linkage S, linkages S1, S2 and S3, as shown in figure 4, in which the identical link-pairs are shown in grey.

For the $6R$ linkages constructed by combining one linkage S and one linkage G through the CLP or CBL method, there are a total of 18 (i.e. $3 \times 3 \times 2$) possible combinations. However, after careful analysis and examination, we find that some of the 18 combinations form identical linkages. Thus, only six distinct types of linkages can be constructed with proposed linkages and methods, which are listed in table 2.

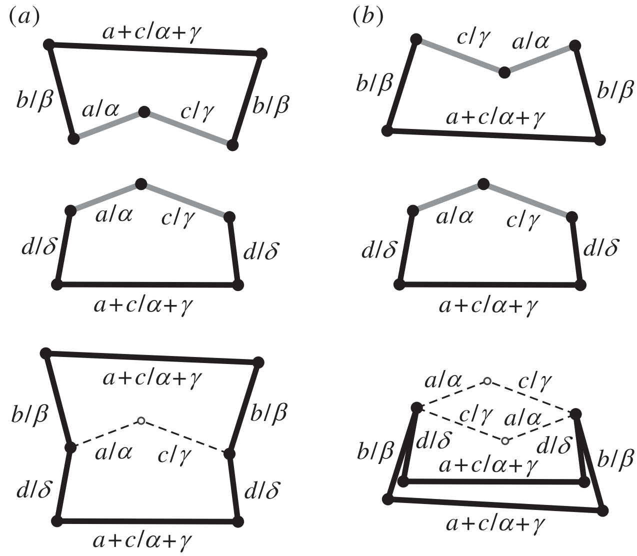


Figure 3. Construction of Wohlhart's double-Goldberg 6R linkage using (a) CLP method; (b) CBL method.

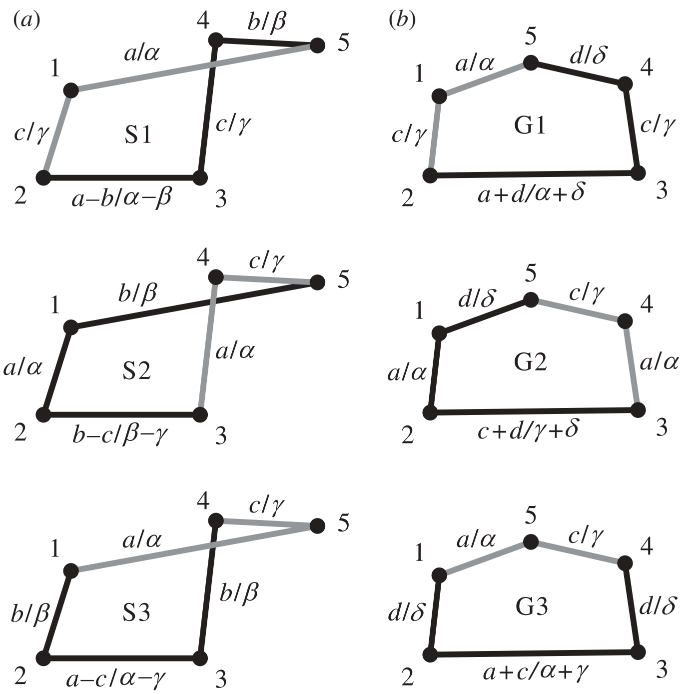
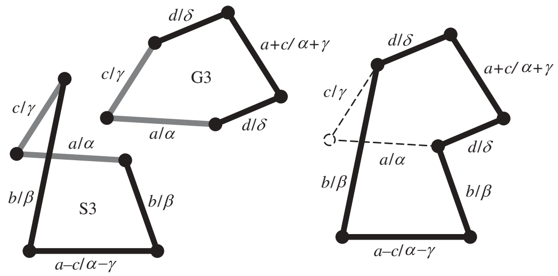


Figure 4. Schematic of the possible linkages S and G with identical link-pair. (a) Linkages S1, S2 and S3; (b) linkages G1, G2 and G3, in which the identical link-pair $a/\alpha \sim c/\gamma$ is marked in grey lines.

Table 2. All possible constructions of the mixed double-Goldberg $6R$ linkages.

<table>
<tr><th rowspan="2">type</th><th colspan="2">linkage</th><th rowspan="2">connection method</th><th rowspan="2">schematic example</th></tr>
<tr><th>S</th><th>G</th></tr>
<tr><td>I</td><td>S3</td><td>G3</td><td>CLP and CBL</td><td>
</td></tr>
<tr><td>II</td><td>S3
S3
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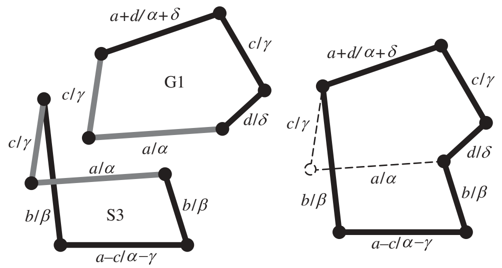
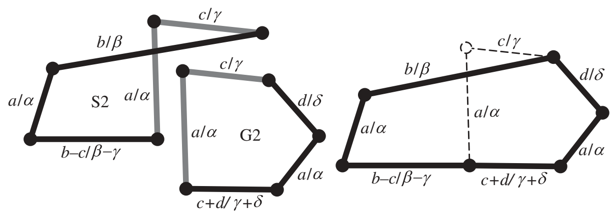
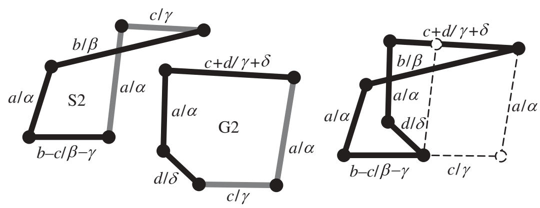
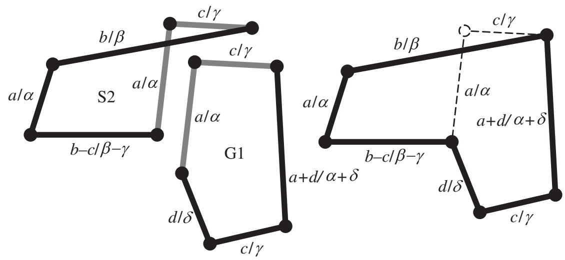
G3</td><td>CLP and CBL</td><td>
</td></tr>
<tr><td>III</td><td>S1
S2</td><td>G1
G2</td><td>CLP</td><td>
</td></tr>
<tr><td>IV</td><td>S1
S2</td><td>G1
G2</td><td>CBL</td><td>
</td></tr>
<tr><td>V</td><td>S1
S2</td><td>G2
G1</td><td>CLP</td><td>
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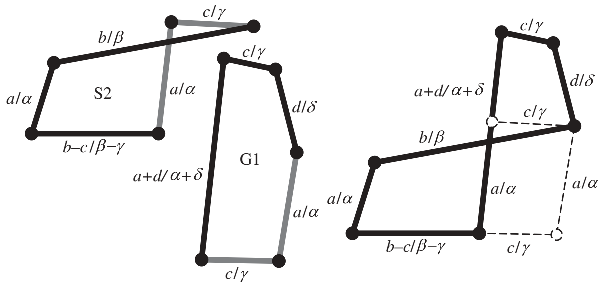
(*Continued.*)

Table 2. (*Continued.*)

type	linkage S	linkage G	connection method	schematic example
VI	S1 S2	G2 G1	CBL	

4. Six types of mixed double-Goldberg 6R linkages

As the six distinct linkages listed in table 2 are built from two different types of 5R linkages, linkages S and G, we name them as a family of mixed double-Goldberg 6R linkages. Here, only the type I linkage is used as an example with detailed derivation of the closure equations, the rest can be obtained similarly. The closure equations, m_1 to m_6 are introduced to shorten the lengthy equations, where

$$m_1 = \frac{\sin((\beta+\alpha)/2)}{\sin((\beta-\alpha)/2)}, \quad m_2 = \frac{\sin((\gamma+\alpha)/2)}{\sin((\gamma-\alpha)/2)}, \quad m_3 = \frac{\sin((\delta+\alpha)/2)}{\sin((\delta-\alpha)/2)},$$
$$m_4 = \frac{\sin((\delta+\gamma)/2)}{\sin((\delta-\gamma)/2)}, \quad m_5 = \frac{\sin((\delta+\beta)/2)}{\sin((\delta-\beta)/2)}, \quad m_6 = \frac{\sin((\beta+\gamma)/2)}{\sin((\beta-\gamma)/2)}. \tag{4.1}$$

(*a*) *Type I mixed double-Goldberg 6R linkage*

Linkages S3 and G3 are selected to build the type I linkage. The identical link-pairs $a/\alpha \sim c/\gamma$ are link-pair 51–45 of both linkages S3 and G3. So, the geometry conditions of linkages S3 and G3 are

$$a_{12}^{\mathrm{S3}} = a_{34}^{\mathrm{S3}} = b, \quad a_{23}^{\mathrm{S3}} = a - c, \quad a_{45}^{\mathrm{S3}} = c, \quad a_{51}^{\mathrm{S3}} = a,$$
$$\alpha_{12}^{\mathrm{S3}} = \alpha_{34}^{\mathrm{S3}} = \beta, \quad \alpha_{23}^{\mathrm{S3}} = \alpha - \gamma, \quad \alpha_{45}^{\mathrm{S3}} = \gamma, \quad \alpha_{51}^{\mathrm{S3}} = \alpha,$$
$$a_{12}^{\mathrm{G3}} = a_{34}^{\mathrm{G3}} = d, \quad a_{23}^{\mathrm{G3}} = a + c, \quad a_{45}^{\mathrm{G3}} = c, \quad a_{51}^{\mathrm{G3}} = a \tag{4.2}$$

and

$$\alpha_{12}^{\mathrm{G3}} = \alpha_{34}^{\mathrm{G3}} = \delta, \quad \alpha_{23}^{\mathrm{G3}} = \alpha + \gamma, \quad \alpha_{45}^{\mathrm{G3}} = \gamma, \quad \alpha_{51}^{\mathrm{G3}} = \alpha.$$

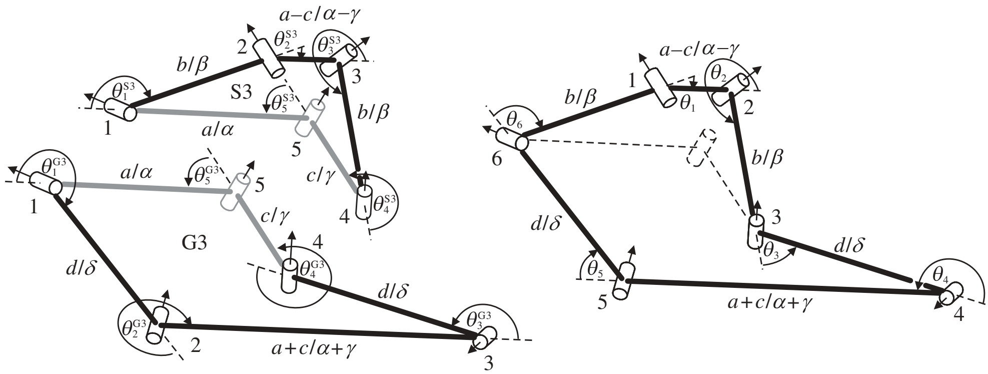


Figure 5. Construction of the type I mixed double-Goldberg $6R$ linkage.

Using the CLP method, a $6R$ linkage can be obtained from linkages S3 and G3 (figure 5). The geometry conditions of the resultant $6R$ linkage are

$$\left.\begin{aligned}
&a_{12}=a-c,\quad a_{23}=a_{61}=b,\quad a_{34}=a_{56}=d,\quad a_{45}=a+c,\\
&\alpha_{12}=\alpha-\gamma,\quad \alpha_{23}=\alpha_{61}=\beta,\quad \alpha_{34}=\alpha_{56}=\delta,\quad \alpha_{45}=\alpha+\gamma,\\
&\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}=\frac{\sin\delta}{d}\\
\text{and}\quad &R_i=0\ (i=1,2,\ldots,6).
\end{aligned}\right\}\tag{4.3}$$

According to equations (2.2) and (2.4), the closure equations of linkages S3 and G3 can be written as

$$\left.\begin{aligned}
&\tan\frac{\theta_2^{\mathrm{S3}}}{2}=-\frac{m_2}{\tan(\theta_1^{\mathrm{S3}}/2)},\quad \tan\frac{\theta_3^{\mathrm{S3}}}{2}=\frac{\tan(\theta_1^{\mathrm{S3}}/2)}{m_6},\\
&\theta_1^{\mathrm{S3}}+\theta_4^{\mathrm{S3}}=2\pi\quad\text{and}\quad \theta_2^{\mathrm{S3}}+\theta_3^{\mathrm{S3}}+\theta_5^{\mathrm{S3}}=2\pi;
\end{aligned}\right\}\tag{4.4}$$

and

$$\left.\begin{aligned}
&\tan\frac{\theta_2^{\mathrm{G3}}}{2}=-\frac{m_2}{\tan(\theta_1^{\mathrm{G3}}/2)},\quad \tan\frac{\theta_3^{\mathrm{G3}}}{2}=m_4\tan\frac{\theta_1^{\mathrm{G3}}}{2},\\
&\theta_1^{\mathrm{G3}}+\theta_4^{\mathrm{G3}}=\pi\quad\text{and}\quad \theta_2^{\mathrm{G3}}+\theta_3^{\mathrm{G3}}+\theta_5^{\mathrm{G3}}=\pi,
\end{aligned}\right\}\tag{4.5}$$

respectively. The relationship between revolute variables of the resultant $6R$ linkage and linkages S3, G3 are

$$\begin{aligned}
&\theta_1=\theta_2^{\mathrm{S3}},\quad \theta_2=\theta_3^{\mathrm{S3}},\quad \theta_3=\theta_4^{\mathrm{S3}}-\theta_4^{\mathrm{G3}}+\pi,\\
&\theta_4=2\pi-\theta_3^{\mathrm{G3}},\quad \theta_5=2\pi-\theta_2^{\mathrm{G3}},\quad \theta_6=\pi+\theta_1^{\mathrm{S3}}-\theta_1^{\mathrm{G3}}.
\end{aligned}\tag{4.6}$$

In addition, the compatibility condition,

$$\theta_5^{\mathrm{S3}} = \theta_5^{\mathrm{G3}}, \tag{4.7}$$

must hold to build a successful connection. Substituting equations (4.6) and (4.7) into equations (4.4) and (4.5) gets the closure equations of the type I linkage,

$$\left.\begin{aligned}
&\tan\frac{\theta_2}{2} = -\frac{m_2}{m_6\tan(\theta_1/2)},\\
&\theta_3 = \pi + 2\tan^{-1}Q - 2\tan^{-1}\left(\frac{m_2}{\tan(\theta_1/2)}\right),\quad \tan\frac{\theta_4}{2} = -\frac{m_2}{Q}\\
\text{and}\quad &\tan\frac{\theta_5}{2} = m_4\tan\frac{\theta_1}{2},\quad \theta_6 = 2\tan^{-1}\left(\frac{m_2}{\tan(\theta_1/2)}\right) - 2\tan^{-1}Q,
\end{aligned}\right\} \tag{4.8}$$

in which

$$Q = \begin{cases}
\dfrac{1}{2H}\left[-\left(\dfrac{1}{m_2}+m_4\right)+\sqrt{\left(\dfrac{1}{m_2}+m_4\right)^2+4H^2\dfrac{m_2}{m_4}}\right] \\
\quad (\theta_1\in[2n\pi,\pi+2n\pi)) \\
\dfrac{1}{2H}\left[-\left(\dfrac{1}{m_2}+m_4\right)-\sqrt{\left(\dfrac{1}{m_2}+m_4\right)^2+4H^2\dfrac{m_2}{m_4}}\right] \\
\quad (\theta_1\in[-\pi+2n\pi,2n\pi))
\end{cases} \tag{4.9}$$

$$\text{and}\qquad H = \frac{m_2 - m_6\tan^2(\theta_1/2)}{(m_2+m_6)\tan(\theta_1/2)}.$$

The input–output curves of the type I linkage are plotted in figure 6.

It should be pointed out that using the CBL method with linkages S3 and G3 in the same configuration, the same 6R linkage can be formed but in a different configuration. In figure 6, the hollow dots represent the configuration of the 6R linkage from the CLP method at $\theta_1 = \theta_1^{\mathrm{CLP}}$, and the solid dots represent the configuration of the 6R linkage from the CBL method at $\theta_1 = \theta_1^{\mathrm{CBL}}$, with $\theta_1^{\mathrm{CLP}} = -\theta_1^{\mathrm{CBL}}$ for all configurations.

(*b*) *Type II mixed double-Goldberg 6R linkage*

Linkages S2 and G3 are selected to build the type II linkage. As shown in figure 7, the identical link-pairs $a/\alpha \sim c/\gamma$ are link-pair 34–45 of linkage S2 and

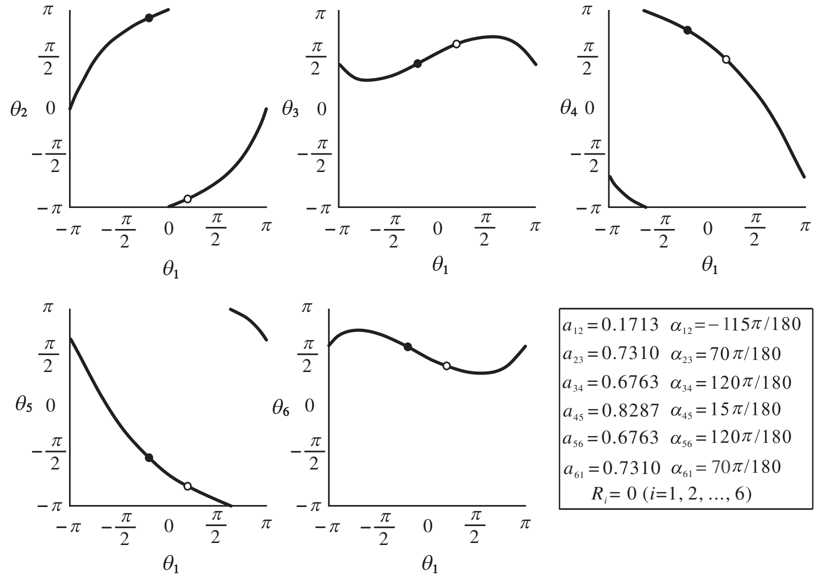


Figure 6. The input–output curves of the type I mixed double-Goldberg $6R$ linkage.

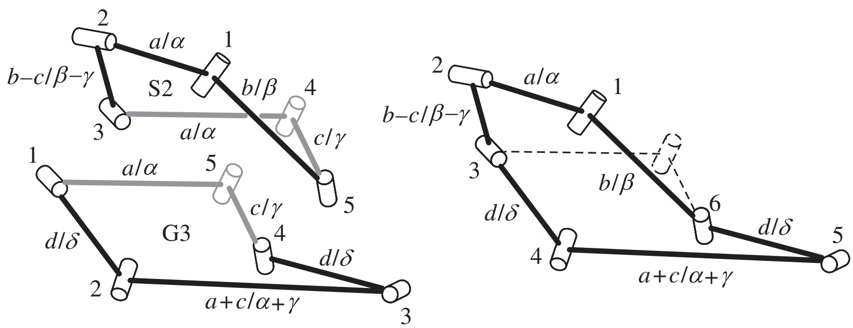


Figure 7. Construction of the type II mixed double-Goldberg $6R$ linkage.

link-pair 51–45 of linkage G3. Then the geometry conditions of linkages S2 and G3 are

$$
\begin{aligned}
&a_{12}^{\mathrm{S2}} = a_{34}^{\mathrm{S2}} = a, \quad a_{23}^{\mathrm{S2}} = b - c, \quad a_{45}^{\mathrm{S2}} = c, \quad a_{51}^{\mathrm{S2}} = b,\\
&\alpha_{12}^{\mathrm{S2}} = \alpha_{34}^{\mathrm{S2}} = \alpha, \quad \alpha_{23}^{\mathrm{S2}} = \beta - \gamma, \quad \alpha_{45}^{\mathrm{S2}} = \gamma, \quad \alpha_{51}^{\mathrm{S2}} = \beta;\\
&a_{12}^{\mathrm{G3}} = a_{34}^{\mathrm{G3}} = d, \quad a_{23}^{\mathrm{G3}} = a + c, \quad a_{45}^{\mathrm{G3}} = c, \quad a_{51}^{\mathrm{G3}} = a,\\
&\alpha_{12}^{\mathrm{G3}} = \alpha_{34}^{\mathrm{G3}} = \delta, \quad \alpha_{23}^{\mathrm{G3}} = \alpha + \gamma, \quad \alpha_{45}^{\mathrm{G3}} = \gamma, \quad \alpha_{51}^{\mathrm{G3}} = \alpha.
\end{aligned}
\tag{4.10}
$$

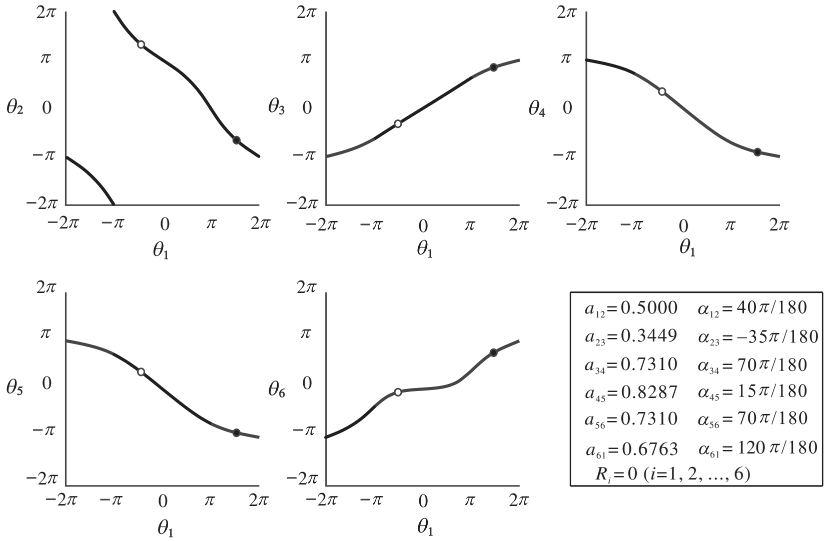


Figure 8. The input–output curves of the type II mixed double-Goldberg 6R linkage.

Using the CLP method with linkages S2 and G3, a 6R linkage can be formed with the geometry conditions as

$$\left.\begin{aligned}
&a_{12}=a,\quad a_{23}=b-c,\quad a_{34}=a_{56}=d,\quad a_{45}=a+c,\quad a_{61}=b,\\
&\alpha_{12}=\alpha,\quad \alpha_{23}=\beta-\gamma,\ \alpha_{34}=\alpha_{56}=\delta,\quad \alpha_{34}=\alpha+\gamma,\quad \alpha_{61}=\beta,\\
&\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}=\frac{\sin\delta}{d}\\
\text{and}\quad &R_i=0\ (i=1,2,\ldots,6).
\end{aligned}\right\}\tag{4.11}$$

The closure equations of the type II linkage can be derived as follows and its input–output curves are plotted in figure 8.

$$\left.\begin{aligned}
&\tan\frac{\theta_2}{2}=\frac{m_1}{\tan(\theta_1/2)},\quad \tan\frac{\theta_3}{2}=\frac{P\cdot\tan(\theta_1/2)-m_2}{\tan(\theta_1/2)+P\cdot m_2},\\
&\tan\frac{\theta_4}{2}=-\frac{m_3}{P},\quad \tan\frac{\theta_5}{2}=-P\cdot m_4\\
\text{and}\quad &\tan\frac{\theta_6}{2}=\frac{\tan^2(\theta_1/2)+m_1m_2-P\cdot(m_1-m_2)\tan(\theta_1/2)}{(m_1-m_2)\tan(\theta_1/2)+P\cdot(m_1m_2+\tan^2(\theta_1/2))},
\end{aligned}\right\}\tag{4.12}$$

in which

$$P=\frac{1}{2m_4\tan(\theta_1/2)}\left[(m_3m_4-1)\pm\sqrt{(m_3m_4-1)^2-4m_3m_4\tan^2\frac{\theta_1}{2}}\right].$$

As shown in table 2, the same linkage can be obtained when using the CBL method. On the input–output curves in figure 8, the linkage obtained from the CLP method (hollow dots) is at the configuration $\theta_1=\theta_1^{\mathrm{CLP}}$, whereas the linkage

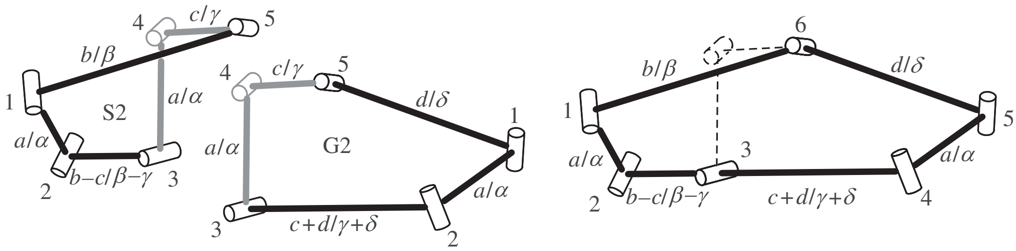


Figure 9. Construction of the type III mixed double-Goldberg 6R linkage.

obtained from the CBL method (solid dots) is at the configuration $\theta_1 = \theta_1^{\rm CBL}$. Then $\theta_1^{\rm CBL} = \theta_1^{\rm CLP} + 2\pi$ is always hold for the same configurations of linkages S2 and G3. It is interesting to note that θ_1 needs to rotate two full circles, or variate in the range of $[-2\pi, 2\pi)$, to make θ_3, θ_4, θ_5 and θ_6 have one full circle of motion, while θ_2 follows with two circles. As given in table 2, the combination between S3 and G1, S3 and G2 or S1 and G3 result in the same 6R linkage.

(c) Types III and IV mixed double-Goldberg 6R linkages

Linkages S2 and G2 are selected to build both types III and IV linkages. Link-pair 34–45 of both linkages are the identical link-pairs. The geometry conditions of linkages S2 and G2 are

$$\begin{aligned}
&a_{12}^{\rm S2} = a_{34}^{\rm S2} = a, \quad a_{23}^{\rm S2} = b - c, \quad a_{45}^{\rm S2} = c, \quad a_{51}^{\rm S2} = b,\\
&\alpha_{12}^{\rm S2} = \alpha_{34}^{\rm S2} = \alpha, \quad \alpha_{23}^{\rm S2} = \beta - \gamma, \quad \alpha_{45}^{\rm S2} = \gamma, \quad \alpha_{51}^{\rm S2} = \beta;\\
&a_{12}^{\rm G2} = a_{34}^{\rm G2} = a, \quad a_{23}^{\rm G2} = c + d, \quad a_{45}^{\rm G2} = c, \quad a_{51}^{\rm G2} = d,\\
&\alpha_{12}^{\rm G2} = \alpha_{34}^{\rm G2} = \alpha, \quad \alpha_{23}^{\rm G2} = \gamma + \delta, \quad \alpha_{45}^{\rm G2} = \gamma, \quad \alpha_{51}^{\rm G2} = \delta.
\end{aligned} \tag{4.13}$$

Using the CLP method, type III linkage can be obtained (figure 9). Then its geometry conditions are

$$\left.\begin{aligned}
&a_{12} = a_{45} = a, \quad a_{23} = b - c, \quad a_{34} = c + d, \quad a_{56} = d, \quad a_{61} = b,\\
&\alpha_{12} = \alpha_{45} = \alpha, \quad \alpha_{23} = \beta - \gamma, \quad \alpha_{34} = \gamma + \delta, \quad \alpha_{56} = \delta, \quad \alpha_{61} = \beta,\\
&\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c} = \frac{\sin\delta}{d}\\
\text{and}\quad &R_i = 0 \ (i = 1, 2, \ldots, 6).
\end{aligned}\right\} \tag{4.14}$$

When using the CBL method with linkages S2 and G2, a type IV linkage is obtained (figure 10). Thus, its geometry conditions are

$$\left.\begin{aligned}
&a_{12} = a_{45} = a, \quad a_{23} = b - c, \quad a_{34} = d, \quad a_{56} = c + d, \quad a_{61} = b,\\
&\alpha_{12} = \alpha_{45} = \alpha, \quad \alpha_{23} = \beta - \gamma, \quad \alpha_{34} = \delta, \quad \alpha_{56} = \gamma + \delta, \quad \alpha_{61} = \beta,\\
&\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c} = \frac{\sin\delta}{d}\\
\text{and}\quad &R_i = 0 \ (i = 1, 2, \ldots, 6).
\end{aligned}\right\} \tag{4.15}$$

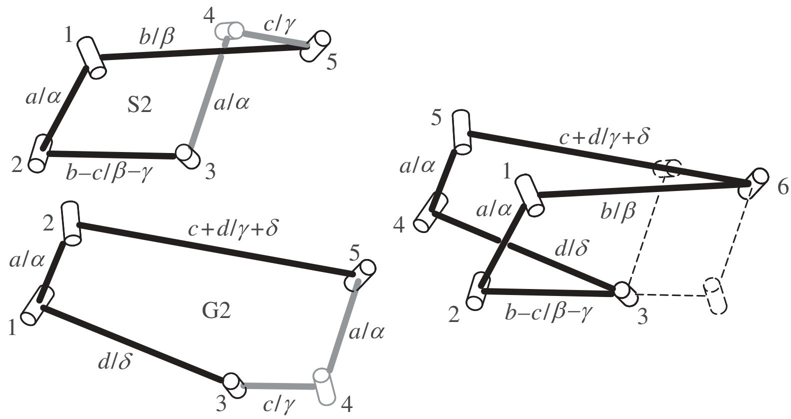


Figure 10. Construction of the type IV mixed double-Goldberg 6R linkage.

The closure equations of type III and IV linkages can be derived as

$$\left.\begin{aligned} &\tan\frac{\theta_2}{2}=\frac{m_1}{\tan(\theta_1/2)},\quad \theta_3=0,\quad \tan\frac{\theta_4}{2}=-m_3\tan\frac{\theta_1}{2},\\ &\theta_5=\pi-\theta_1\quad\text{and}\quad \tan\frac{\theta_6}{2}=\frac{(1-m_1m_3)\tan(\theta_1/2)}{m_1+m_3\tan^2(\theta_1/2)} \end{aligned}\right\}\tag{4.16}$$

and

$$\left.\begin{aligned} &\tan\frac{\theta_2}{2}=\frac{m_1}{\tan(\theta_1/2)},\quad \tan\frac{\theta_3}{2}=\frac{m_2+m_3\tan^2(\theta_1/2)}{(m_2m_3-1)\tan(\theta_1/2)},\quad \theta_4=\pi-\theta_1,\\ &\tan\frac{\theta_5}{2}=m_3\tan\frac{\theta_1}{2}\quad\text{and}\quad \tan\frac{\theta_6}{2}=\frac{m_1m_2+\tan^2(\theta_1/2)}{(m_1-m_2)\tan(\theta_1/2)}, \end{aligned}\right\}\tag{4.17}$$

respectively. Their input–output curves are plotted in figures 11 and 12. For the type III linkage, it is obvious that θ_3 is constrained to zero during the full circle of motion. The instantaneous mobility of joint 3 is locked to zero, i.e. the link-pair 23–34 could be viewed as a composite link 24 of $b+d/\beta+\delta$. Therefore, the type III linkage is equivalent to a Goldberg 5R linkage. As demonstrated in table 2, the connection of linkages S1 and G1 gives the same 6R linkages.

(*d*) *Types V and VI mixed double-Goldberg 6R linkages*

Linkages S2 and G1 are selected to build both types V and VI linkages. Link-pair 34–45 of linkage S2 and link-pair 51–12 of linkage G1 are the identical

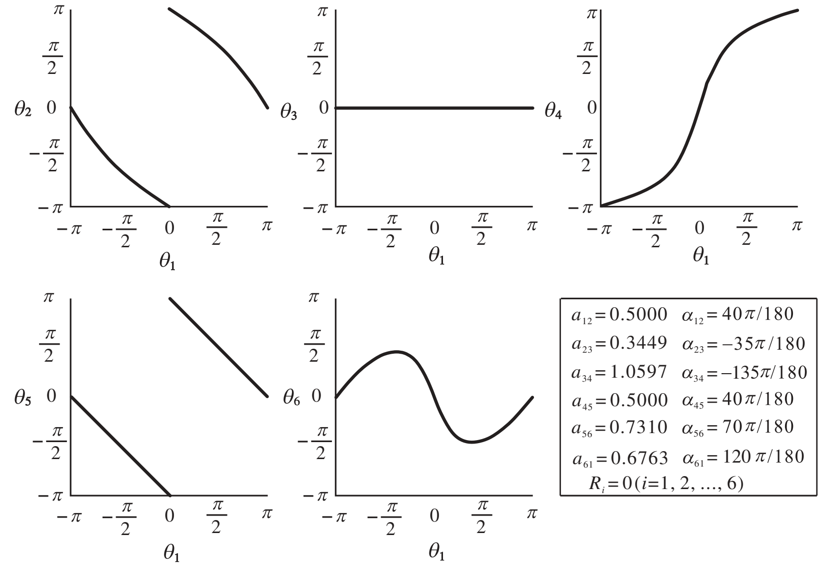


Figure 11. The input–output curves of the type III mixed double-Goldberg $6R$ linkage.

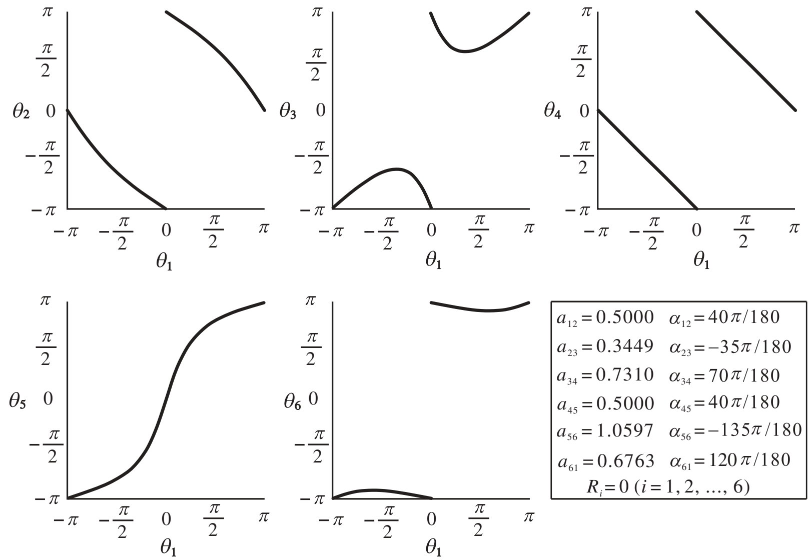


Figure 12. The input–output curves of the type IV mixed double-Goldberg $6R$ linkage.

link-pairs. The geometry conditions of linkages S2 and G1 are

$$\left.\begin{aligned}
&a_{12}^{\mathrm{S2}} = a_{34}^{\mathrm{S2}} = a, \quad a_{23}^{\mathrm{S2}} = b - c, \quad a_{45}^{\mathrm{S2}} = c, \quad a_{51}^{\mathrm{S2}} = b,\\
&\alpha_{12}^{\mathrm{S2}} = \alpha_{34}^{\mathrm{S2}} = \alpha, \quad \alpha_{23}^{\mathrm{S2}} = \beta - \gamma, \quad \alpha_{45}^{\mathrm{S2}} = \gamma, \quad \alpha_{51}^{\mathrm{S2}} = \beta;\\
&a_{12}^{\mathrm{G1}} = a_{34}^{\mathrm{G1}} = c, \quad a_{23}^{\mathrm{G1}} = a + d, \quad a_{45}^{\mathrm{G1}} = d, \quad a_{51}^{\mathrm{G1}} = a,\\
&\alpha_{12}^{\mathrm{G1}} = \alpha_{34}^{\mathrm{G1}} = \gamma, \quad \alpha_{23}^{\mathrm{G1}} = \alpha + \delta, \quad \alpha_{45}^{\mathrm{G1}} = \delta, \quad \alpha_{51}^{\mathrm{G1}} = \alpha.
\end{aligned}\right\} \tag{4.18}$$

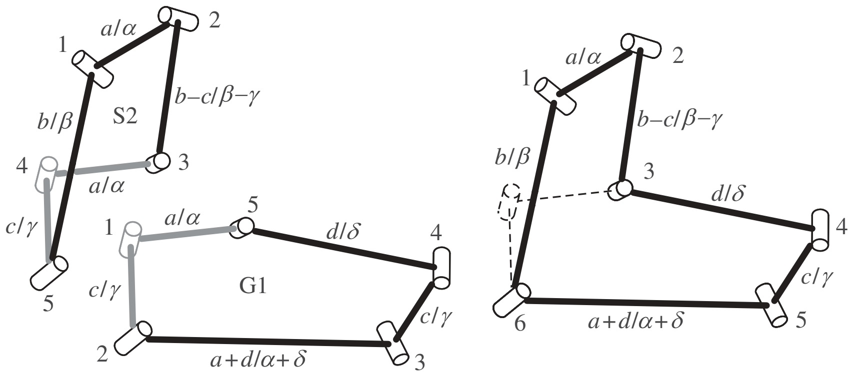


Figure 13. Construction of the type V mixed double-Goldberg $6R$ linkage.

Using the CLP method a type V linkage is obtained, as shown in figure 13. Its geometry conditions are

$$\left.\begin{aligned} &a_{12}=a,\quad a_{23}=b-c,\quad a_{34}=d,\quad a_{45}=c,\quad a_{56}=a+d,\quad a_{61}=b,\\ &\alpha_{12}=\alpha,\quad \alpha_{23}=\beta-\gamma,\quad \alpha_{34}=\delta,\quad \alpha_{45}=\gamma,\quad \alpha_{56}=\alpha+\delta,\quad \alpha_{61}=\beta,\\ &\qquad\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}=\frac{\sin\delta}{d}\\ \text{and}\quad &\qquad R_i=0\ (i=1,2,\ldots,6). \end{aligned}\right\} \tag{4.19}$$

Similar to types III and IV, a type VI linkage can be obtained when using the CBL method with linkages S2 and G1 (figure 14). The geometry conditions of the type VI linkage are

$$\left.\begin{aligned} &a_{12}=a,\quad a_{23}=b-c,\quad a_{34}=a+d,\quad a_{45}=c,\quad a_{56}=d,\quad a_{61}=b,\\ &\alpha_{12}=\alpha,\quad \alpha_{23}=\beta-\gamma,\quad \alpha_{34}=\alpha+\delta,\quad \alpha_{45}=\gamma,\quad \alpha_{56}=\delta,\quad \alpha_{61}=\beta,\\ &\qquad\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}=\frac{\sin\delta}{d}\\ \text{and}\quad &\qquad R_i=0\ (i=1,2,\ldots,6). \end{aligned}\right\} \tag{4.20}$$

The closure equations of the types V and VI linkages can be derived as

$$\left.\begin{aligned} &\tan\frac{\theta_2}{2}=\frac{m_1}{\tan(\theta_1/2)},\quad \tan\frac{\theta_3}{2}=\frac{1}{m_4\tan(\theta_1/2)},\\ &\theta_4=\pi-\theta_1,\theta_5=\pi-\theta_3\quad\text{and}\quad\tan\frac{\theta_6}{2}=-\frac{m_1}{\tan(\theta_1/2)} \end{aligned}\right\} \tag{4.21}$$

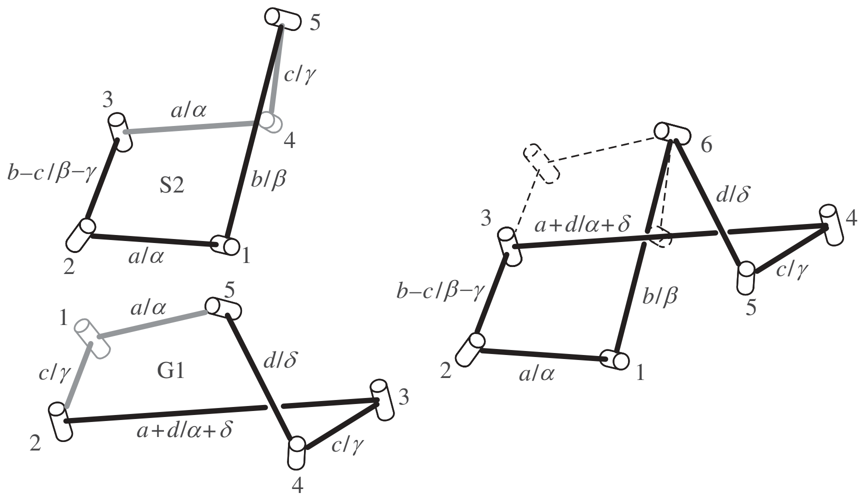


Figure 14. Construction of the type VI mixed double-Goldberg 6R linkage.

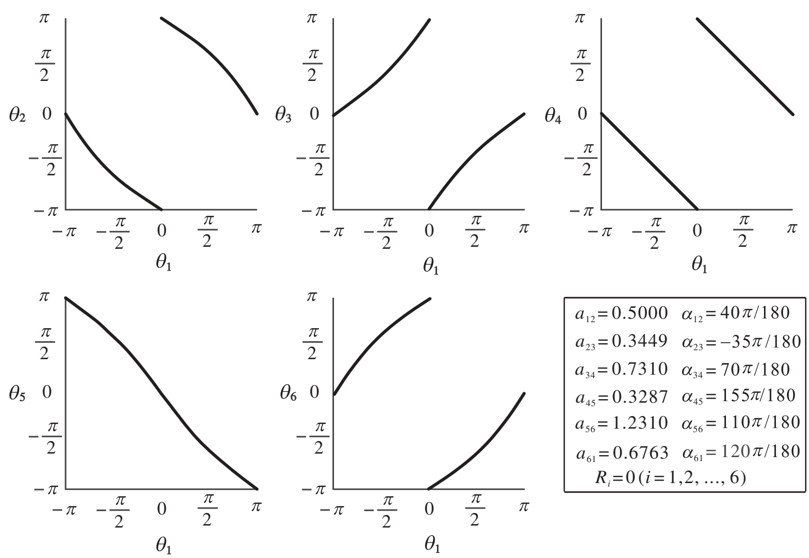


Figure 15. The input–output curves of the type V mixed double-Goldberg 6R linkage.

and

$$\left.\begin{aligned}
&\tan\frac{\theta_2}{2}=\frac{m_1}{\tan(\theta_1/2)},\quad \tan\frac{\theta_3}{2}=\frac{\tan(\theta_1/2)}{m_2},\\
&\tan\frac{\theta_4}{2}=m_4\tan\frac{\theta_1}{2},\quad \theta_5=\pi-\theta_1\quad\text{and}\\
&\tan\frac{\theta_6}{2}=\frac{(m_4\tan^2(\theta_1/2)+m_1m_2m_4+m_1-m_2)\tan(\theta_1/2)}{(m_1m_4-m_2m_4-1)\tan^2(\theta_1/2)-m_1m_2};
\end{aligned}\right\}\tag{4.22}$$

respectively. Their input–output curves are plotted in figures 15 and 16.

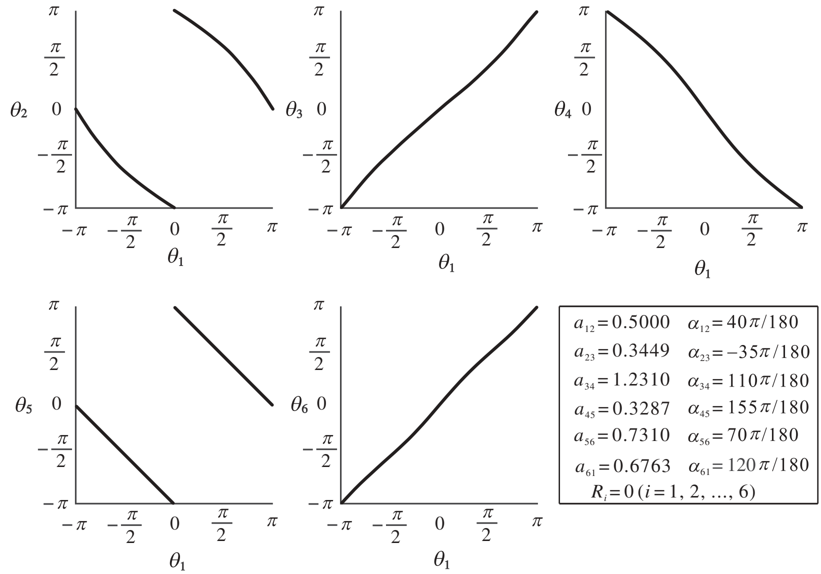


Figure 16. The input–output curves of the type VI mixed double-Goldberg 6R linkage.

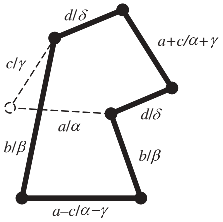
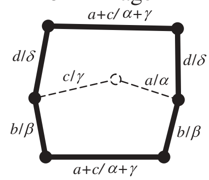
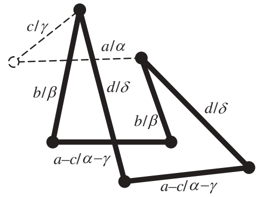
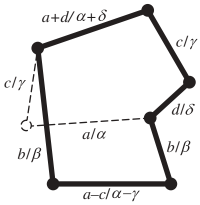
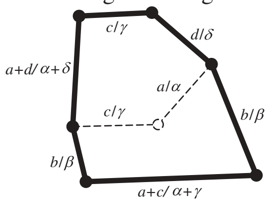
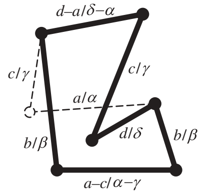
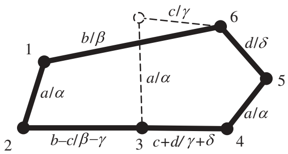
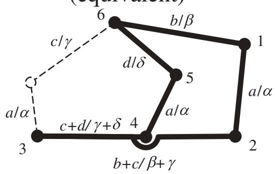
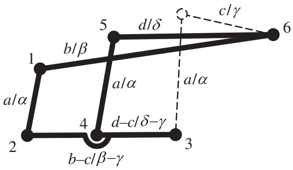
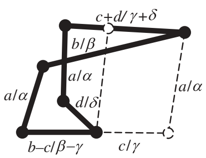
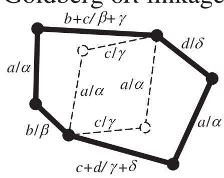
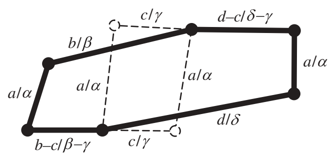
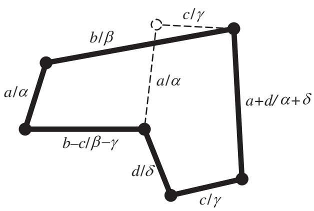
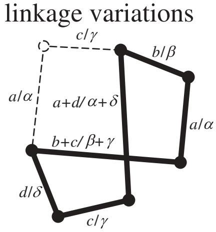
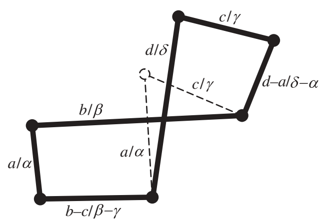
Figure 17. The motion sequence of the type V linkage model.

From the geometry conditions, the type V linkage is in fact a Goldberg 6R linkage constructed from one Bennett linkage, with links a/α, b/β and another Bennett linkage with links c/γ, d/δ connected side by side. It can also be considered as a special case of Waldron's hybrid 6R linkage (Waldron 1968). The type VI linkage is related to the L-shape Goldberg 6R linkage (Goldberg 1943) where link 23 in the resultant 6R linkage is negative in length (Chen & You 2005, 2008). Moreover, both 6R linkages can also be constructed with linkages S1 and G2, as listed in table 2.

5. Conclusion and discussions

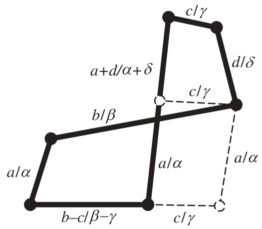
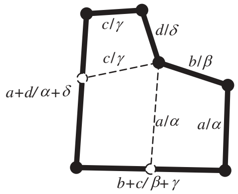
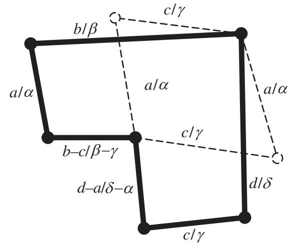
A new family of mixed double-Goldberg 6R linkages have been built by the combination of a subtractive Goldberg 5R linkage and a Goldberg 5R linkage through the CLP or CBL method. All of them have a single degree of freedom. The physical model of the type V linkage is shown in figure 17. From the

Table 3. The complete families of double-Goldberg 6*R* linkages.

type	the mixed double-Goldberg linkage from linkages S and G	the double-Goldberg linkage from two G linkages	the subtractive double-Goldberg linkage from two S linkages
I		Wohlhart's double-Goldberg 6*R* linkage 	
II		Baker's first variant of Goldberg 6*R* linkage 	
III		Goldberg 5*R* linkage (equivalent) 	
IV		Baker's second variant of Goldberg 6*R* linkage 	
V		one of Goldberg's 6*R* linkage variations 	

(*Continued.*)

Table 3. (*Continued.*)

<table>
<tr><th>type</th><th>the mixed double-Goldberg linkage from linkages S and G</th><th>the double-Goldberg linkage from two G linkages</th><th>the subtractive double-Goldberg linkage from two S linkages</th></tr>
<tr><td>VI</td><td>
</td><td>The L-shape Goldberg 6R linkage

</td><td>
</td></tr>
</table>

construction process and geometry conditions, the type I, II, IV and VI linkages can be regarded as new linkages, whereas the type III linkage is equivalent to the Goldberg 5R linkage, the type V linkage is the 6R linkage from two Bennett linkages demonstrated in Goldberg's original paper (Goldberg 1943).

In this article, we used two different types of Goldberg 5R linkages as construction elements. Alternatively, two Goldberg 5R linkages or two subtractive Goldberg 5R linkages can be used to construct 6R linkages through the CLP or CBL method. The resultant linkages are the double-Goldberg 6R linkages reported previously and the corresponding subtractive cases, respectively, as listed in table 3. Owing to the identical Bennett ratio in links a/α, b/β and d/δ, type I of the double-Goldberg 6R linkages from two linkages G, i.e. Wohlhart's double-Goldberg linkage, can be transformed into the line-symmetric Bricard linkage through isomerization; so does the type I 6R linkage from two linkages S. Thus, such linkages with implicit symmetric characteristics exhibit the similar bifurcation behaviours as the special line-symmetric Bricard linkage without offset. However, the type I mixed double-Goldberg 6R linkage has no such problem as it generally has no explicit or implicit symmetric properties.

All linkages in the family of mixed double-Goldberg 6R linkages are built from four basic links a/α, b/β, c/γ and d/δ. After comparing the geometry conditions of each linkage type, an *extra link* c/γ is identified, which plays a role different to that of the other three links (a/α, b/β and d/δ). In type V and VI linkages, link c/γ is one of the six individual links in the linkage, while for the rest, link c/γ does not directly exist in the linkage but hides in links, such as $a+c/\alpha+\gamma$, $b-c/\beta-\gamma$, etc. For example, in the type I linkage, when c/γ is shrunk to zero, links 12 and 45 will be link a/α at the same time. The type I linkage then becomes a Mavroidis and Roth 6R linkage with zero offsets; similar observations can be found in other linkage types. When link c/γ is changed to a negative length, linkages S and G are swapped with each other. The resultant 6R linkages are still the mixed double-Goldberg 6R linkages.

By now, it can be concluded that all possible 6R overconstrained linkages constructed from two Goldberg 5R linkages of the same kind, or two different kinds, through the CLP or the CBL method, have been found and listed in this article. The relationships among all of the Bennett-based $6R$ linkages that are involved have been disclosed. This work paves the road for the application of these $6R$ linkages in the design of reconfigurable mechanisms (Zhang & Dai 2009).

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