Mechanism and Machine Theory 46 (2011) 1097–1106

Contents lists available at ScienceDirect

Mechanism and Machine Theory

journal homepage: www.elsevier.com/locate/mechmt

A spatial 6*R* linkage derived from subtractive Goldberg 5*R* linkages

C.Y. Song, Y. Chen *

School of Mechanical and Aerospace Engineering, Nanyang Technological University, 50 Nanyang Avenue, Singapore 639798, Singapore

ARTICLE INFO

Article history:
Received 24 August 2010
received in revised form 19 March 2011
accepted 19 March 2011
Available online 14 April 2011

Keywords:
Overconstrained linkage
Bennett linkage
Goldberg 5*R* linkage
Subtractive Goldberg 5*R* linkage
Double subtractive Goldberg 6*R* linkage

ABSTRACT

In this paper, a subtractive Goldberg 5*R* linkage is defined as a variation of Goldberg 5*R* linkage. A spatial 6*R* linkage is constructed by combining two subtractive Goldberg 5*R* linkages through a common Bennett linkage. This 6*R* linkage, namely double subtractive Goldberg 6*R* linkage, appears to be distinct from other existing spatial 6*R* overconstrained linkages reported before. Both the overconstrained geometric conditions and the closure equations of the proposed linkage are derived. Physical models are also made to validate the linkage.

1. Introduction

Overconstrained linkages are a family of special linkages with full-range mobility even though they do not obey the Grübler–Kutzbach mobility criterion [1]. Their mobility is due to the special geometric conditions among the links and joint axes that are called overconstrained conditions. One of the most famous and impressive 3D overconstrained linkages is the Bennett linkage, a spatial 4*R* linkage [2,3]. Since its publication in 1903, it has received an enormous attention from kinematicians. A number of 3D overconstrained linkages have been found by combining or merging several Bennett linkages, such as Myard linkage [4], Goldberg linkage [5], Bennett-joint 6*R* linkage [6], Dietmaier's 6*R* linkage [7], Wohlhart's double-Goldberg linkage [8], some cases of Waldron's hybrid 6*R* linkage [9], and so on. All these linkages can be classified into a family of Bennett-based linkages. Among them, Goldberg's constructing method sets an example to build new overconstrained linkages. The Goldberg 5*R* or 6*R* linkages can be formed by firstly superposing two linkages, either a Bennett linkage or a Goldberg linkage, then removing the common link or joint which shares the same spatial geometry, and finally rigidifying the angular relationships between two adjacent links connected to the removed common joint. In the constructing point of view, Myard linkage was commonly considered as a special case of Goldberg 5*R* linkage [10]. Later, Wohlhart derived a more general case of Goldberg 5*R* linkage and proposed a 6*R* linkage from two Goldberg 5*R* linkages combined face to face [8]. Recently, another 6*R* linkage has been constructed by combining two Goldberg 5*R* linkages in a back-to-back configuration [11]. A throughout and in-depth research on the most of the Bennett-based linkages was given by [12]. In this paper, we intend to combine two subtractive Goldberg 5*R* linkages to build a different type of spatial 6*R* linkage through a common Bennett linkage.

The layout of this paper is as follows. In Section 2, the Bennett linkage and the Goldberg 5*R* linkage are reviewed. Then the geometric conditions and the closure equations for the subtractive Goldberg 5*R* linkage are derived. Section 3 presents the procedure to build the double subtractive Goldberg 6*R* linkage in detail. In Section 4, the characteristics and extensions of double subtractive Goldberg 6*R* linkages are discussed, which concludes the paper.

* Corresponding author. Tel.: +65 6790 5941; fax: +65 6790 4062.
E-mail address: chenyan@ntu.edu.sg (Y. Chen).

0094-114X/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.mechmachtheory.2011.03.006

Notations

z_i	coordinate axis along the revolute axis of joint i;
x_i	coordinate axis along the common normal from z_{i-1} to z_i;
$a_{(i-1)i}$	length of link $(i-1)i$, which is the common normal distance from z_{i-1} to z_i positively about x_i;
$\alpha_{(i-1)i}$	twist of link $(i-1)i$, which is the rotation angle from z_{i-1} to z_i positively about x_i;
R_i	offset of joint i, which is the common normal distance from x_i to x_{i+1} positively along z_i;
θ_i	revolute variable of joint i, which is the rotation angle from x_i to x_{i+1} positively about z_i;
a/α	the link with link length a and twist α;
L, R, A, B	superscripts that denote different linkages during constructing process.

2. Bennett linkage and Goldberg 5*R* linkage

2.1. Bennett linkage

The Bennett linkage is the only overconstrained 4*R* linkage with the axes of four revolute joints neither parallel nor concurrent [2]. The geometric parameters and closure equations of the Bennett linkage in Fig. 1 are

$$\begin{aligned} a_{12} &= a_{34} = a, a_{23} = a_{41} = b, \\ \alpha_{12} &= \alpha_{34} = \alpha, \alpha_{23} = \alpha_{41} = \beta, \\ \frac{\sin\alpha}{a} &= \frac{\sin\beta}{b}, \\ R_i &= 0 (i = 1,\ 2,\ 3\ \text{and}\ 4); \end{aligned} \tag{1}$$

and

$$\begin{aligned} &\theta_1 + \theta_3 = 2\pi, \theta_2 + \theta_4 = 2\pi, \\ &\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = \frac{\sin\dfrac{\beta+\alpha}{2}}{\sin\dfrac{\beta-\alpha}{2}}, \end{aligned} \tag{2}$$

respectively.

2.2. Goldberg 5*R* linkage

The Goldberg 5*R* linkage [5] is obtained by combining a pair of Bennett linkages in such a way that the link and joint common to both are removed and a pair of adjacent links are rigidly attached to each other as shown in Fig. 2. The techniques developed by Goldberg can be summarised as the summation of two Bennett linkages to produce a 5*R* linkage, or the subtraction of a primary composite linkage from another Bennett linkage to form a syncopated linkage. Here we only consider the summation case of Goldberg 5*R* linkage.

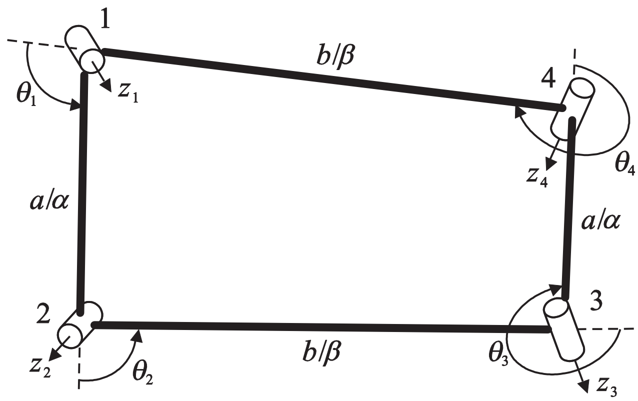

Fig. 1. The Bennett linkage.

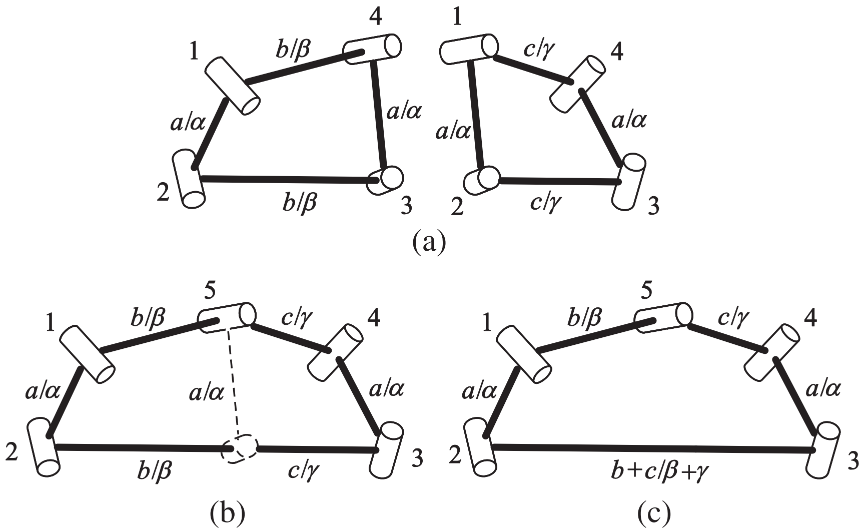

Fig. 2. The construction of the Goldberg 5*R* linkage. (a) Two Bennett linkages with the common link *a*/*α* in the middle; (b) the two links at the bottom are rigidified to be collinear; (c) the Goldberg 5*R* linkage is formed after removing the common link and joint.

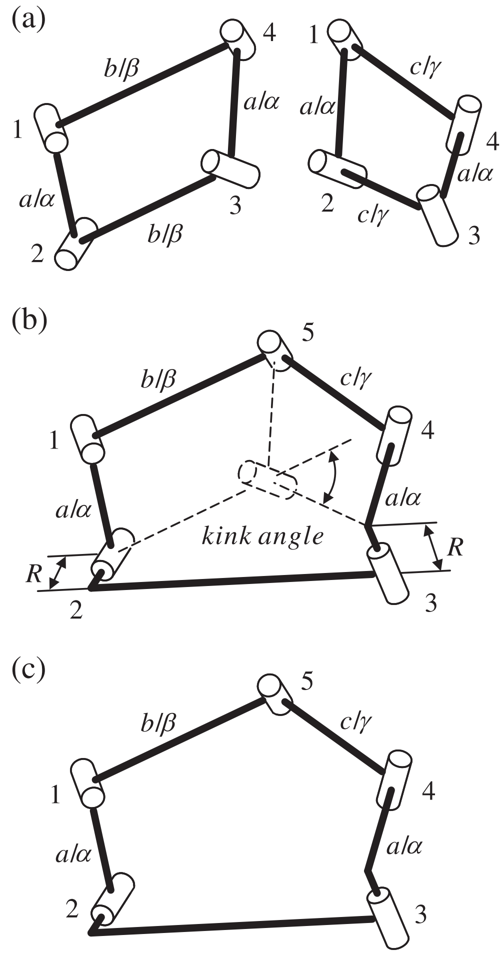

Fig. 3. The construction of the generalised Goldberg 5*R* linkage. (a) Two Bennett linkages are firstly superposed with the common link *a*/*α* in the middle; (b) then kink angle is set to be any value. The new link is added to connect two joints at the bottom, two offsets appear correspondingly; (c) last, after removing three links and one joint inside, the generalised Goldberg 5*R* linkage is obtained.

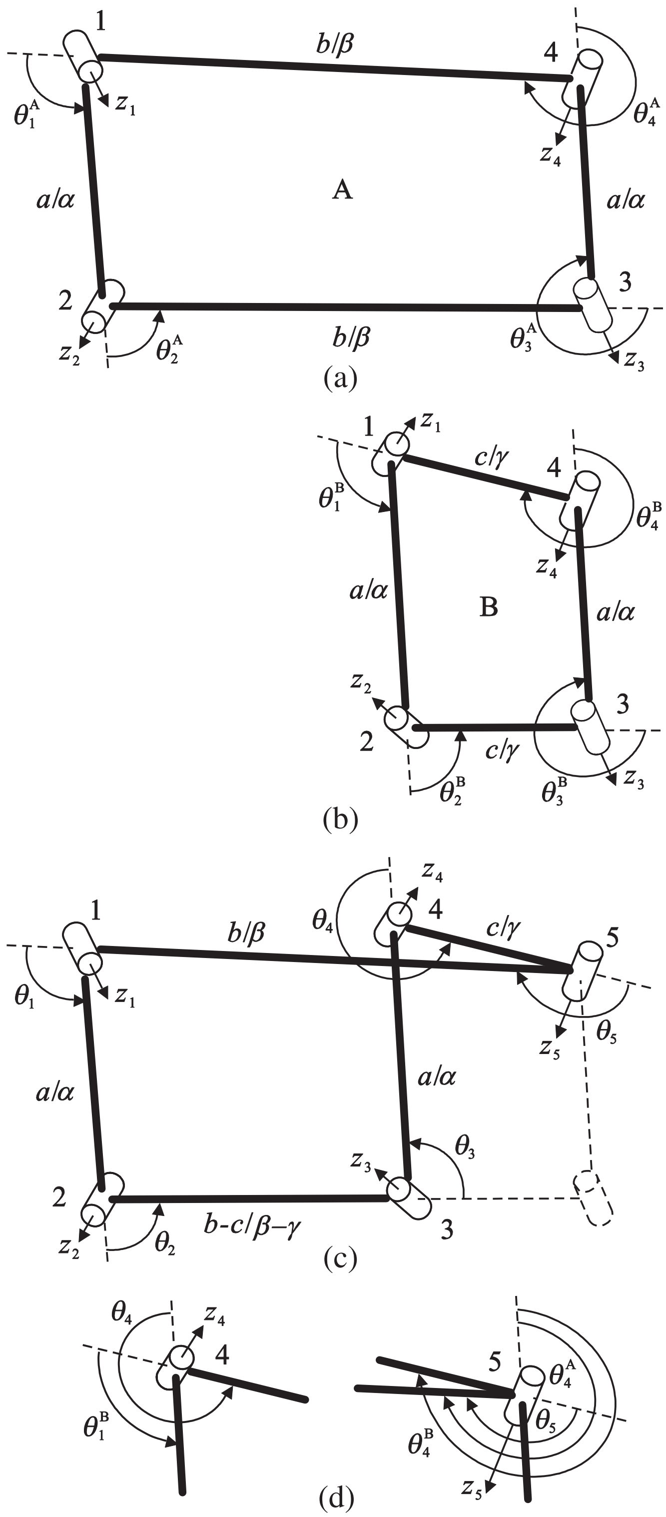

Fig. 4. The construction of the subtractive Goldberg 5*R* linkage. (a) and (b) Two Bennett linkages with one common link *a*/*α*; (c) the subtractive Goldberg 5*R* linkage; (d) the detailed relationship between the kinematic variables at joints 4 and 5.

A more generalised 5*R* linkage was also proposed by Goldberg briefly [5] and later derived in detail by Wohlhart [8]. In the general case, two links which form the rigidified link are not collinearly posed. A variable “kink angle” was introduced, see Fig. 3. Therefore, Goldberg 5*R* linkage is a special case of the generalised 5*R* linkage when the kink angle is zero.

2.3. Subtractive Goldberg 5R linkage

When the kink angle in the generalised Goldberg 5R linkage equals to π, the two links adjacent to the common link are overlapped and the resultant linkage is in fact formed by subtracting Bennett linkage B from Bennett linkage A as shown in Fig. 4, which is the first variation of the Goldberg linkages given in [5]. Therefore, we call such linkage as the *subtractive Goldberg 5R linkage*.

Consider two Bennett linkages shown in Figs. 4(a) and (b). In order to construct the subtractive Goldberg 5R linkage, these Bennett linkages must have a common link with the identical geometric parameters, a/α. So the Bennett linkages A and B should have the following geometric parameters.

$$\begin{gathered} a_{12}^{A} = a_{34}^{A} = a_{12}^{B} = a_{34}^{B} = a, a_{23}^{A} = a_{41}^{A} = b, a_{23}^{B} = a_{41}^{B} = c, \\ \alpha_{12}^{A} = \alpha_{34}^{A} = \alpha_{12}^{B} = \alpha_{34}^{B} = \alpha, \alpha_{23}^{A} = \alpha_{41}^{A} = \beta, \alpha_{23}^{B} = \alpha_{41}^{B} = \gamma, \\ \frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}, \\ R_i^A = R_i^B = 0 (i = 1,\ 2,\ 3\ \text{and}\ 4). \end{gathered} \tag{3}$$

Then the closure equations of Bennett linkages A and B are

$$\begin{gathered} \theta_1^A + \theta_3^A = 2\pi, \theta_2^A + \theta_4^A = 2\pi, \\ \tan\frac{\theta_1^A}{2}\tan\frac{\theta_2^A}{2} = \frac{\sin\dfrac{\beta+\alpha}{2}}{\sin\dfrac{\beta-\alpha}{2}}; \end{gathered} \tag{4}$$

and

$$\begin{gathered} \theta_1^B + \theta_3^B = 2\pi, \theta_2^B + \theta_4^B = 2\pi, \\ \tan\frac{\theta_1^B}{2}\tan\frac{\theta_2^B}{2} = \frac{\sin\dfrac{\gamma+\alpha}{2}}{\sin\dfrac{\gamma-\alpha}{2}}, \end{gathered} \tag{5}$$

respectively.

The subtractive Goldberg 5R linkage can be formed by removing the common links and joint as shown in Fig. 4(c). The geometric parameters of the subtractive Goldberg 5R linkage are

$$\begin{gathered} a_{12} = a_{34} = a, a_{23} = b-c, a_{45} = c, a_{51} = b, \\ \alpha_{12} = \alpha_{34} = \alpha, \alpha_{23} = \beta-\gamma, \alpha_{45} = \gamma, \alpha_{51} = \beta, \\ \frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}, \\ R_i = 0 (i = 1,\ 2, \ldots,\ 5). \end{gathered} \tag{6}$$

In Fig. 4, the relationship among the revolute variables of the subtractive Goldberg 5R linkage and the Bennett linkages A and B can be set as follows.

$$\begin{gathered} \theta_1 = \theta_1^A, \theta_2 = \theta_2^A, \theta_3 = \pi-\theta_2^B, \\ \theta_4 = 2\pi-\theta_1^B, \theta_5 = \pi-\left(\theta_4^B-\theta_4^A\right). \end{gathered} \tag{7}$$

And in order to construct the subtractive Goldberg 5R linkage, the compatibility between Bennett linkages A and B,

$$\theta_3^A = \theta_3^B, \tag{8}$$

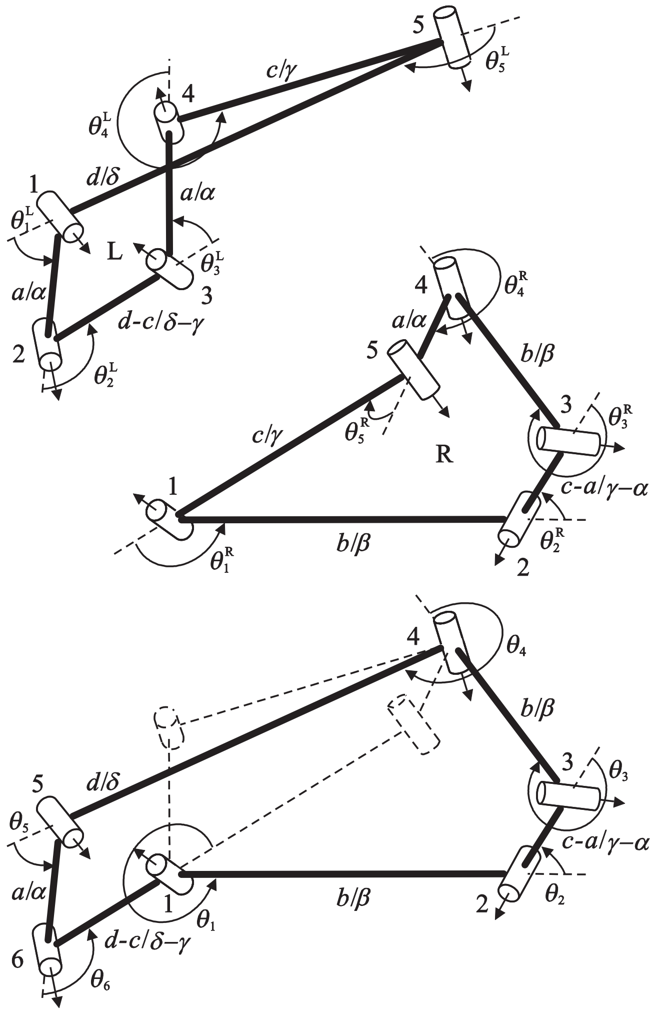

Fig. 5. The construction of a 6*R* linkage from two subtractive Goldberg 5*R* linkages by removing the common Bennett linkage in dash lines.

must be preserved. Substituting Eqs. (7) and (8) into Eqs. (4) and (5), the closure equations of the subtractive Goldberg 5*R* linkage can be derived as follows.

$$\begin{gathered}\theta_1+\theta_4=2\pi, \theta_2+\theta_3+\theta_5=2\pi, \\ \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}=\frac{\sin\frac{\beta+\alpha}{2}}{\sin\frac{\beta-\alpha}{2}}, \text{and} \frac{\tan\frac{\theta_1}{2}}{\tan\frac{\theta_3}{2}}=\frac{\sin\frac{\gamma+\alpha}{2}}{\sin\frac{\gamma-\alpha}{2}}.\end{gathered} \tag{9}$$

3. Double subtractive Goldberg 6*R* linkage

Fig. 4 shows that in the subtractive Goldberg 5*R* linkage, the link-pairs 51–12 and 34–45 are from Bennett linkages A and B, respectively. Therefore, we can build 6*R* linkages from two subtractive Goldberg 5*R* linkages that share a common Bennett linkage A or B, which is similar as the construction of Goldberg 6*R* linkages [5] and their variations [12]. Alternatively, due to the geometric condition that $\sin \alpha/a = \sin \gamma/c$ in Eq. (3), we can use link-pair 45–51 to form another common Bennett linkage, which is different from Bennett linkage A or B.

For two subtractive Goldberg 5R linkages with the following geometric parameters,

$$\begin{aligned} a_{12}^{L} = a_{34}^{L} = a, a_{23}^{L} = d - c, a_{45}^{L} = c, a_{51}^{L} = d, \\ \alpha_{12}^{L} = \alpha_{34}^{L} = \alpha, \alpha_{23}^{L} = \delta - \gamma, \alpha_{45}^{L} = \gamma, \alpha_{51}^{L} = \delta; \\ a_{12}^{R} = a_{34}^{R} = b, a_{23}^{R} = c - a, a_{45}^{R} = a, a_{51}^{R} = c, \\ \alpha_{12}^{R} = \alpha_{34}^{R} = \beta, \alpha_{23}^{R} = \gamma - \alpha, \alpha_{45}^{R} = \alpha, \alpha_{51}^{R} = \gamma, \end{aligned} \tag{10}$$

the link-pair 34–45 of linkage L and link-pair 45–51 of linkage R are the same as a/α and c/γ. Therefore, a 6R linkage can be obtained by constructing a common Bennett linkage from these two link-pairs, see in Fig. 5. Thus, the geometry conditions of this 6R linkage are

$$\begin{aligned} a_{12} = a_{34} = b, a_{23} = c - a, a_{45} = d, a_{56} = a, a_{61} = d - c, \\ \alpha_{12} = \alpha_{34} = \beta, \alpha_{23} = \gamma - \alpha, \alpha_{45} = \delta, \alpha_{56} = \alpha, \alpha_{61} = \delta - \gamma, \\ \frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c} = \frac{\sin\delta}{d}, \\ R_i = 0 (i = 1, 2, \ldots, 6). \end{aligned} \tag{11}$$

According to Eq. (9), the closure equations of the subtractive Goldberg 5R linkages L and R are

$$\begin{aligned} \theta_1^{L} + \theta_4^{L} = 2\pi, \theta_2^{L} + \theta_3^{L} + \theta_5^{L} = 2\pi, \\ \tan\frac{\theta_1^{L}}{2}\tan\frac{\theta_2^{L}}{2} = \frac{\sin\frac{\delta+\alpha}{2}}{\sin\frac{\delta-\alpha}{2}} = m_1, \frac{\tan\frac{\theta_1^{L}}{2}}{\tan\frac{\theta_3^{L}}{2}} = \frac{\sin\frac{\gamma+\alpha}{2}}{\sin\frac{\gamma-\alpha}{2}} = m_2; \end{aligned} \tag{12}$$

and

$$\begin{aligned} \theta_1^{R} + \theta_4^{R} = 2\pi, \theta_2^{R} + \theta_3^{R} + \theta_5^{R} = 2\pi, \\ \tan\frac{\theta_1^{R}}{2}\tan\frac{\theta_2^{R}}{2} = \frac{\sin\frac{\gamma+\beta}{2}}{\sin\frac{\gamma-\beta}{2}} = m_3, \frac{\tan\frac{\theta_1^{R}}{2}}{\tan\frac{\theta_3^{R}}{2}} = \frac{\sin\frac{\alpha+\beta}{2}}{\sin\frac{\alpha-\beta}{2}} = m_4, \end{aligned} \tag{13}$$

respectively.

Here $m_i (i = 1, 2, 3$ and $4)$ are set to simplify the representations of different relationships of twists for later derivation. The relationship between the revolute variables of the objective 6R linkage and the subtractive Goldberg 5R linkages L and R are

$$\begin{aligned} \theta_1 = \theta_1^{R} - \pi, \theta_2 = \theta_2^{R}, \theta_3 = \theta_3^{R}, \\ \theta_4 = \left(\theta_3^{L} - \pi\right) + \left(\theta_5^{L} - \pi\right) - \left(\pi - \theta_4^{R}\right) = \theta_3^{L} + \theta_5^{L} + \theta_4^{R} - 3\pi, \\ \theta_5 = \theta_1^{L}, \theta_6 = \theta_2^{L}. \end{aligned} \tag{14}$$

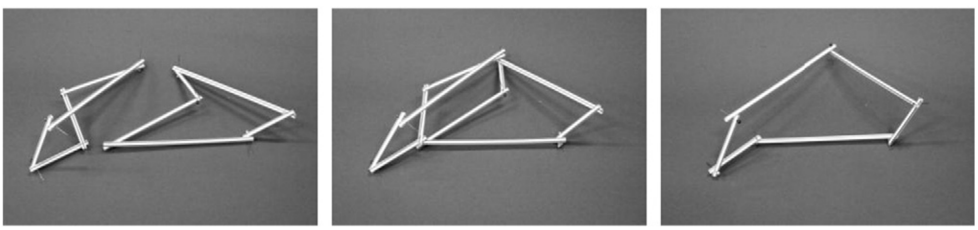

Fig. 6. The constructing process of the double subtractive Goldberg 6R linkage in physical models.

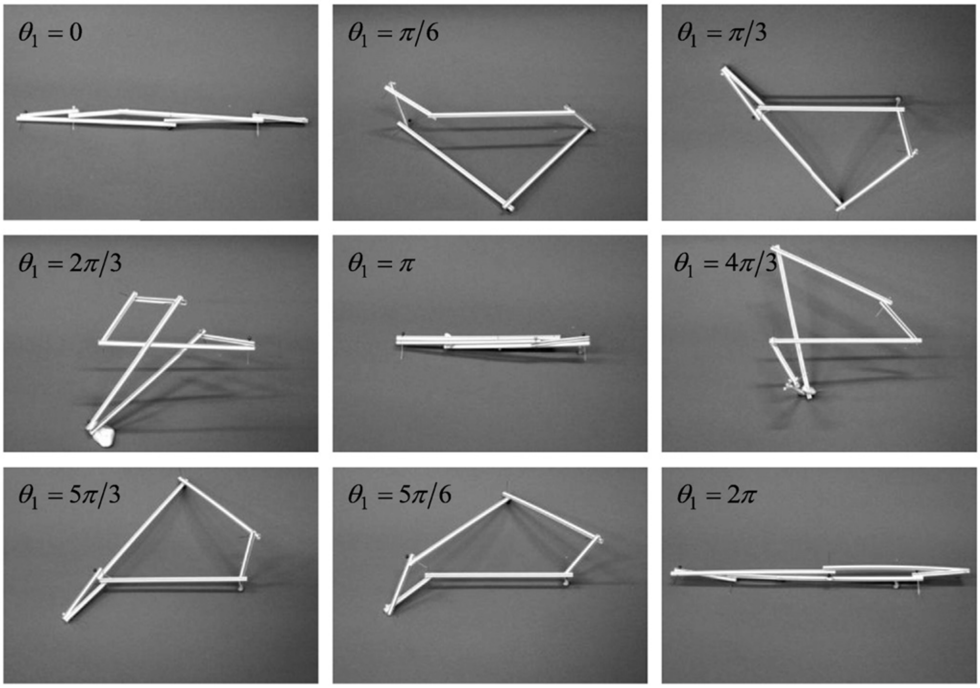

Fig. 7. The full circle movement of the double subtractive Goldberg 6*R* linkage.

For the common Bennett linkage shared by these two subtractive Goldberg 5*R* linkages, as shown in dash lines in Fig. 5, the compatibility relationship,

$$\theta_4^L + \theta_5^R = 2\pi, \tag{15}$$

should be preserved.

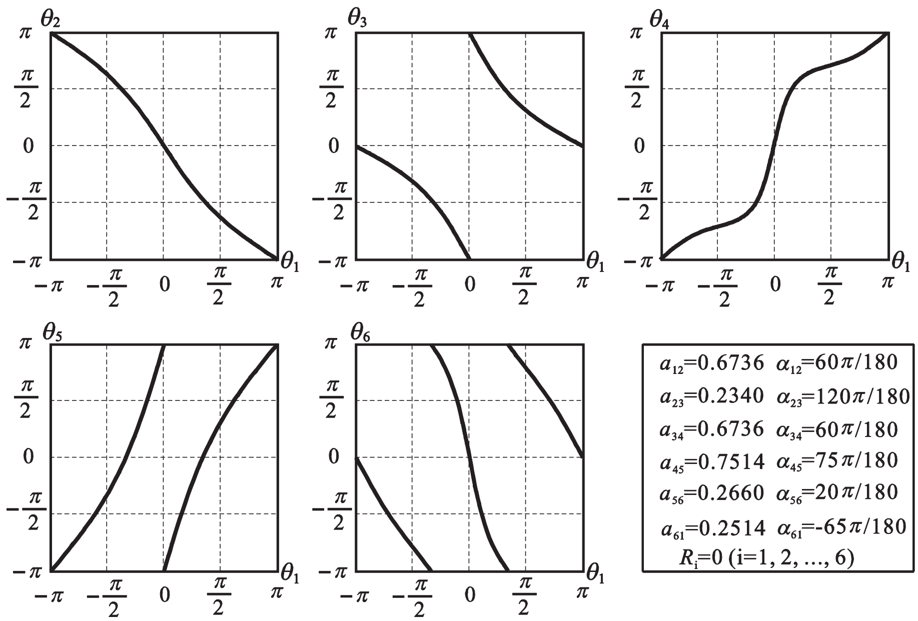

Fig. 8. The input–output curves of the linkage, θ_1 vs. $\theta_i (i = 2, 3, \ldots, 6)$.

Table 1
Variations of link length c and geometric parameters of the 6R linkage ($d>a$).

Variation of c	Geometric parameters	
	Changed	Unchanged
$c>d>a$	$a_{23}=c-a,\ \alpha_{23}=\gamma-\alpha$ $a_{61}=c-d,\ \alpha_{61}=\gamma-\delta$	$a_{12}=a_{34}=b,\ \alpha_{12}=\alpha_{34}=\beta$ $a_{34}=d,\ \alpha_{34}=\delta$ $a_{56}=a,\ \alpha_{56}=\alpha$ $R_i=0(i=1,\ 2,\dots,\ 6)$
$d>c>a$	$a_{23}=c-a,\ \alpha_{23}=\gamma-\alpha$ $a_{61}=d-c,\ \alpha_{61}=\delta-\gamma$	
$d>a>c$	$a_{23}=a-c,\ \alpha_{23}=\alpha-\gamma$ $a_{61}=d-c,\ \alpha_{61}=\delta-\gamma$	
$c=0$	$a_{23}=a,\ \alpha_{23}=\alpha$ $a_{61}=d,\ \alpha_{61}=\delta$	
$c<0$	$a_{23}=a-c,\ \alpha_{23}=\alpha-\gamma$ $a_{61}=d-c,\ \alpha_{61}=\delta-\gamma$	

Substituting Eqs. (14) and (15) into Eqs. (12) and (13), we can derive the closure equations of this 6R linkage as follows,

$$\tan\frac{\theta_3}{2}=-\frac{1}{m_4\tan\frac{\theta_1}{2}} \tag{16}$$

which show that the newly formed 6R linkage has mobility one. Because of the construct method, it can be called as the *double subtractive Goldberg 6R linkage*. Physical models are also made to validate this linkage. Fig. 6 is the constructing process of the double subtractive Goldberg 6R linkage shown in physical models. Fig. 7 shows the full circle movement of the constructed linkage, whose input–output curves are presented in Fig. 8.

4. Conclusions and discussion

In this paper, a subtractive Goldberg 5R linkage has been defined following the similar method of constructing the original Goldberg 5R linkage, which can also be considered as a special case of the generalised Goldberg 5R linkage. Such two linkages have been combined together to form a spatial 6R linkage by removing the common Bennett linkage. Thus this 6R linkage is called double subtractive Goldberg 6R linkage. The geometric conditions and the closure equations have been derived in detail and the linkage has mobility one. From both the constructing method and the overconstrained geometric conditions, it is indicated that the proposed linkage is different from all the existing spatial 6R overconstrained linkages.

From the geometric parameters in Eq. (11), it is interesting to notice that in the six links of the double subtractive Goldberg 6R linkage, links 12, 34, 45 and 56 are determined by links a/α, b/β or d/δ individually, while links 23 and 61 are determined by $c-a/\gamma-\alpha$ and $d-c/\delta-\gamma$. Although link c/γ does not appear in the final 6R linkage, as it has been removed with the common Bennett linkage, it still plays an important role in the overall construction of the 6R linkage. This characteristic is only available for the double subtractive Goldberg 6R linkage.

When the relationship between links a/α, c/γ and d/δ varies, other linkages, which are similar to the double subtractive Goldberg linkage, can be derived. Table 1 lists the geometric parameters for the variation of c comparing to a and d. Here, we assume that link length $d>a$, as similar conclusions can be drawn for $a>d$. When $c>d>a$, $d>c>a$ or $d>a>c$, three variations of the double subtractive Goldberg 6R linkage can be formed. All have mobility one. When $c=0$, the two subtractive Goldberg 5R linkages that build up the 6R linkage are shrunk into two Bennett linkages. The mobility of the resultant 6R linkage is raised to two. Especially, when $c<0$, the geometric parameters are in the same setup as the case of $d>a>c$. And the 6R linkage is also a variation of the proposed 6R linkage with mobility one. However, physically the subtractive Goldberg 5R linkage is transformed into the original Goldberg 5R linkage and the resultant 6R linkage is the second variant of L-shaped Goldberg 6R linkage proposed by Baker [12].

These variations of the *extra link c/γ* could be explained by the concept of "negative length", which is to describe possible mobile assemblies of the Bennett linkages [13,14]. In our case, the term "negative" indicates that the direction of synthesis is opposite to each other. If we allow the length to be negative, and then these variations are basically belong to same linkage type when the parameters on the extra link vary.

Furthermore, we can apply the same combination method as the generalised Goldberg 5R linkage [8] to form a more general case of the double Goldberg 6R linkage when the kink angle falls in the range of $[0,\ 2\pi)$. In this case, offset values will not be all zeros. This is one possible method to introduce non-zero offset to the new 6R linkage. However, due to the complexity of the math calculation, the general formulas will not be derived here.

Acknowledgement

Y. Chen would like to acknowledge the support from the Nanyang Technological University (NTU), Singapore, in the form of a research grant (RG10/09). C. Y. Song would like to thank NTU for providing the University Graduate Scholarship during his PhD study.

References

[1] K.H. Hunt, Kinematic Geometry of Mechanisms, Clarendon Press, 1990.
[2] G.T. Bennett, A new mechanism, Engineering 76 (1903) 777–778.
[3] G.T. Bennett, The skew isogram mechanism, Proceedings of the London Mathematical Society 2 (1914) 151.
[4] F.E. Myard, Contribution à la géométrie des systèmes articulés, Bulletin de la Société Mathématique de France 59 (1931) 183–210.
[5] M. Goldberg, New five-bar and six-bar linkages in three dimensions, Transactions of the ASME 65 (1943) 649–663.
[6] C. Mavroidis, B. Roth, New and revised overconstrained mechanisms, Journal of Mechanical Design 117 (1995) 75–82.
[7] P. Dietmaier, A new spatial 6R linkage, Proceedings 9th World Congress IFToMM, 1, 1995, pp. 52–56.
[8] K. Wohlhart, Merging two general goldberg 5*R* linkages to obtain a new 6*R* space mechanism, Mechanism and Machine Theory 26 (1991) 659–668.
[9] K.J. Waldron, Hybrid overconstrained linkages, Journal of Mechanisms 3 (1968) 73–78.
[10] J.E. Baker, The Bennett, Goldberg and Myard linkages—in perspective, Mechanism and Machine Theory 14 (1979) 239–253.
[11] Y. Chen, Z. You, Spatial 6*R* linkages based on the combination of two Goldberg 5*R* linkages, Mechanism and Machine Theory 42 (2007) 1484–1498.
[12] J.E. Baker, A comparative survey of the Bennett-based, 6-revolute kinematic loops, Mechanism and Machine Theory 28 (1993) 83–96.
[13] Y. Chen, Z. You, Mobile assemblies based on the Bennett linkage, Proceedings of the Royal Society A (Mathematical, Physical and Engineering Sciences) 461 (2005) 1229–1245.
[14] Y. Chen, Z. You, On mobile assemblies of Bennett linkages, Proceedings of the Royal Society A (Mathematical, Physical and Engineering Sciences) 464 (2008) 1275–1283.