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THE ORIGINAL DOUBLE-GOLDBERG 6*R* LINKAGE AND ITS BIFURCATION ANALYSIS

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Abstract. *In this paper, the original double-Goldberg 6R linkage is obtained by merging two Goldberg 5R linkages on the commonly shared "roof-links" and then removing this connection to achieve the linkage. Two forms of the 6R linkage are obtained from this constructive method. Bifurcation analysis is performed to the original double-Goldberg 6R linkage to identify two other forms of the linkage in a full-circle movement. Transformation among these linkage forms reveals interesting morphing property of the linkage, which is worth attentions from related researchers.*

Notations:

z_i : coordinate axis along the revolute axis of joint i ;

x_i : coordinate axis along the common normal from z_{i-1} to z_i ;

$a_{(i-1)i}$: length of link $(i-1)i$, which is the common normal distance from z_{i-1} to z_i positively about x_i ;

$\alpha_{(i-1)i}$: twist of link $(i-1)i$, which is the rotation angle from z_{i-1} to z_i positively about x_i ;

R_i : offset of joint i , which is the common normal distance from x_i to x_{i+1} positively along z_i ;

θ_i : revolute variable of joint i , which is the rotation angle from x_i to x_{i+1} positively about z_i ;

a/α , b/β , c/γ , d/δ : the corresponding link length and twist of the same link are a and α , b and β , c and γ , or d and δ , respectively;

L, R: superscripts that denote different linkages during construction process.

1 INTRODUCTION

The study of overconstrained mechanisms has always been an interesting topic in mechanism design for over a century. Researchers such as Bennett [1, 2] and Bricard [3] have set the foundations and proposed their basic designs of overconstrained linkages. Most of the latter researchers are using existing linkages as the building block to create new overconstrained linkages. Goldberg [4] was the pioneer who used two or three Bennett linkages to build a series of 5*R* and 6*R* linkages by merging them on the common links or subtracting a linkage out of a primary loop. Later, Waldron [5] found a hybrid linkage by merging two Bennett linkages on a commonly shared joint axis to achieve a single-loop overconstrained 6*R* linkage. Wohlhart [6] further developed this concept by using two generalized Goldberg 5*R* linkages to be merged on the common “roof-links” to form an overconstrained 6*R* linkage. Chen & You [7] used similar method to combine two original Goldberg 5*R* linkages in a back-to-back configuration to form another overconstrained 6*R* linkage. Recently, Song and Chen [8] used one subtractive Goldberg 5*R* linkage and one original Goldberg 5*R* linkage to build a series of overconstrained linkages. In the area of kinematic analysis, Baker [9, 10] did the most work in studying linkages that using the Bennett linkages as the building block. However, little work has been done about linkages that are made of 5*R* linkages. In this paper, an original double-Goldberg 6*R* linkage, which is made of two original Goldberg 5*R* linkages, is studied at length. The bifurcation behavior of the linkage is investigated in detail to reveal the morphing nature of this particular linkage.

This paper is presented as follows. Section 2 introduces the kinematics of the original Goldberg 5*R* linkage. The two constructive forms of the original double-Goldberg 6*R* linkages are derived in section 3. In section 4, bifurcation analysis is performed to identify two non-constructive forms of the linkage. Conclusion is enclosed in section 5, which ends the paper.

2 KINEMATICS OF THE ORIGINAL GOLDBERG 5*R* LINKAGE

When two Bennett linkages are merged on a common link and two adjacent links are linearly posed, after removing the common link and joint, a Goldberg 5*R* linkage will be obtained as shown in Fig. (1).

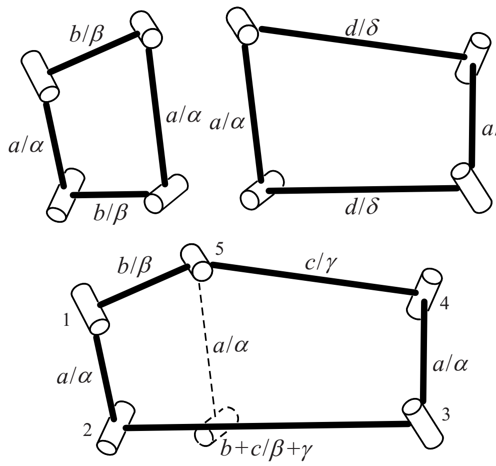

Figure 1: Construction of a Goldberg 5*R* linkage.

The geometry conditions and closure equations of the linkage are listed as

$$a_{12}=a_{34},\ a_{23}=a_{45}+a_{51},$$
$$\alpha_{12}=\alpha_{34},\ \alpha_{23}=\alpha_{45}+\alpha_{51},$$
$$\frac{\sin\alpha_{12}}{a_{12}}=\frac{\sin\alpha_{45}}{a_{45}}=\frac{\sin\alpha_{51}}{a_{51}}, \tag{1}$$
$$R_i=0\ (i=1,2,...,5);$$

and

$$\tan\frac{\theta_2}{2}=\frac{\sin\frac{\alpha_{51}+\alpha_{12}}{2}}{\tan\frac{\theta_1}{2}\sin\frac{\alpha_{51}-\alpha_{12}}{2}},\ \tan\frac{\theta_3}{2}=\frac{\tan\frac{\theta_1}{2}\sin\frac{\alpha_{45}+\alpha_{12}}{2}}{\sin\frac{\alpha_{45}-\alpha_{12}}{2}},\ \theta_1+\theta_4=\pi,$$
$$\tan\frac{\theta_5}{2}=\frac{\left(1-\frac{\sin\frac{\alpha_{51}+\alpha_{12}}{2}}{\sin\frac{\alpha_{51}-\alpha_{12}}{2}}\frac{\sin\frac{\alpha_{45}+\alpha_{12}}{2}}{\sin\frac{\alpha_{45}-\alpha_{12}}{2}}\right)\tan\frac{\theta_1}{2}}{\frac{\sin\frac{\alpha_{51}+\alpha_{12}}{2}}{\sin\frac{\alpha_{51}-\alpha_{12}}{2}}+\frac{\sin\frac{\alpha_{45}+\alpha_{12}}{2}}{\sin\frac{\alpha_{45}-\alpha_{12}}{2}}\tan^2\frac{\theta_1}{2}}. \tag{2}$$
$$(\text{or } \theta_2+\theta_3+\theta_5=\pi)$$

respectively. From the relationship between the revolute variables shown in Eq. (2), when we perform the algebraic substitution to make $x_i=\tan\frac{\theta_i}{2}\ (i=1,2,...,6)$, it is obvious that x_5 is quadratically related to x_1, while $x_{2,3,4}$ are linearly related to x_1. Therefore, for each value of θ_5, we can find two configurations of the linkage on the kinematic paths. For example, as shown in Fig. (2), when $\theta_5=5\pi/9$, we can locate configurations L1 and L2.

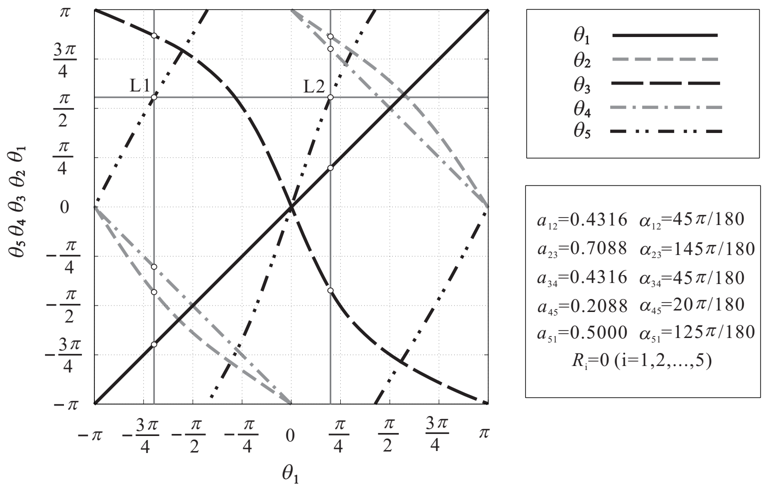

Figure 2: The input-output curves of the original Goldberg 5*R* linkage and the two configurations L1 and L2 when $\theta_5 = 5\pi/9$.

3 THE ORIGINAL DOUBLE-GOLDBERG 6*R* LINKAGE

In order to build the original double-Goldberg 6*R* linkage, two Goldberg 5*R* linkages, namely linkages L and R, are such prepared so that they commonly share the same geometry conditions on the "roof-links", or link-pair 45-51. Therefore, we can have the geometry conditions of both linkages as

$$\begin{gathered} a_{12}^{\mathrm{L}} = a_{34}^{\mathrm{L}} = b\,,\ a_{23}^{\mathrm{L}} = a + c\,,\ a_{45}^{\mathrm{L}} = a\,,\ a_{51}^{\mathrm{L}} = c\,, \\ \alpha_{12}^{\mathrm{L}} = \alpha_{34}^{\mathrm{L}} = \beta\,,\ \alpha_{23}^{\mathrm{L}} = \alpha + \gamma\,,\ \alpha_{45}^{\mathrm{L}} = \alpha\,,\ \alpha_{51}^{\mathrm{L}} = \gamma\,; \\ a_{12}^{\mathrm{R}} = a_{34}^{\mathrm{R}} = d\,,\ a_{23}^{\mathrm{R}} = a + c\,,\ a_{45}^{\mathrm{R}} = a\,,\ a_{51}^{\mathrm{R}} = c\,, \\ \alpha_{12}^{\mathrm{R}} = \alpha_{34}^{\mathrm{R}} = \delta\,,\ \alpha_{23}^{\mathrm{R}} = \alpha + \gamma\,,\ \alpha_{45}^{\mathrm{R}} = \alpha\,,\ \alpha_{51}^{\mathrm{R}} = \gamma\,. \end{gathered} \tag{3}$$

From the results in previous section, take the configuration when $\theta_5^{\mathrm{L}} = 5\pi/9$ for example, we can find two layouts of linkage L, as shown in Fig. (3). Similarly, we can also find two layouts of linkage R when $\theta_5^{\mathrm{R}} = 5\pi/9$, as shown in Fig. (4). These configurations of linkages L and R will be used to build the original double-Goldberg 6*R* linkage.

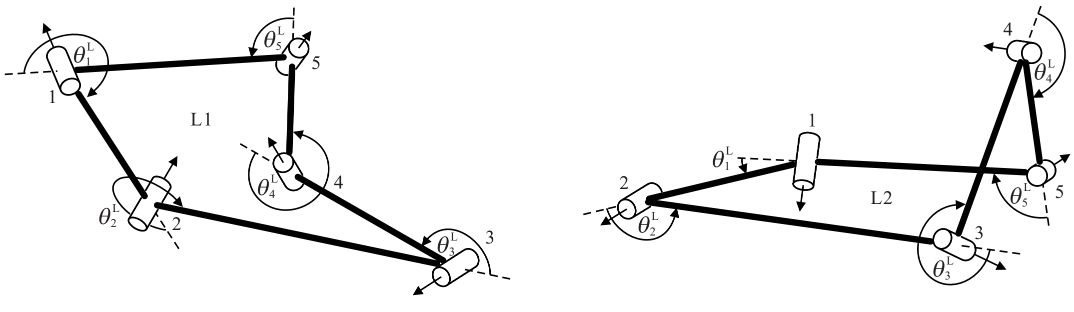

Figure 3: The spatial layout of L1 and L2 when $\theta_5^{\mathrm{L}} = 5\pi/9$.

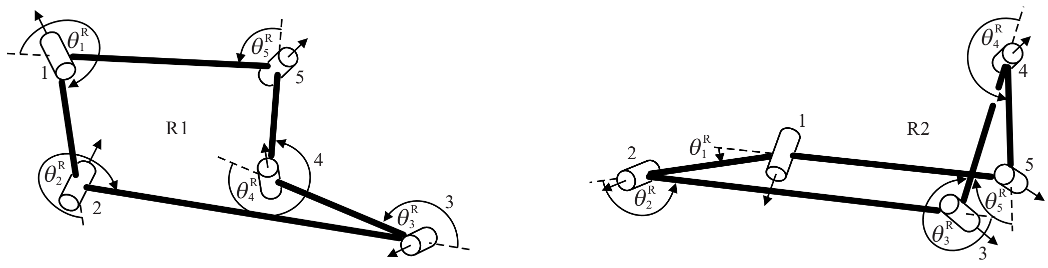

Figure 4: The spatial layout of R1 and R2 when $\theta_5^{\mathrm{R}} = 5\pi/9$.

3.1 Form I of the Original Double-Goldberg 6*R* Linkage

Linkages L1 and R2 will be used to build the Form I linkage. We first merge them on the commonly shared link-pair 45-51. Then, by removing this common link-pair, a single-loop overconstrained 6*R* linkage will be achieved as shown in Fig. (5). The geometry conditions of the Form I linkage are

$$\begin{gathered}
a_{12} = a_{45} = a + c,\ a_{23} = a_{61} = b,\ a_{34} = a_{56} = d, \\
\alpha_{12} = \alpha_{45} = \alpha + \gamma,\ \alpha_{23} = \alpha_{61} = \beta,\ \alpha_{34} = \alpha_{56} = \delta, \\
\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c} = \frac{\sin\delta}{d}, \\
R_i = 0\ (i = 1, 2, \ldots, 6).
\end{gathered} \tag{4}$$

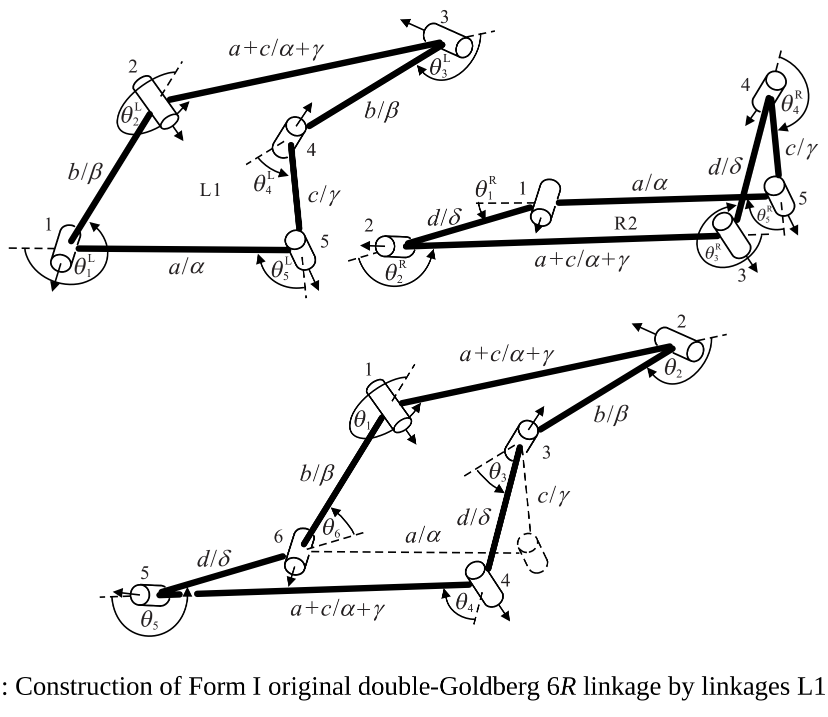

Figure 5: Construction of Form I original double-Goldberg 6*R* linkage by linkages L1 and R2.

According to Eq. (2), the closure equations of linkages L and R are

$$\tan\frac{\theta_2^{\mathrm{L}}}{2}=-\frac{m_1}{\tan\frac{\theta_1^{\mathrm{L}}}{2}},\ \tan\frac{\theta_3^{\mathrm{L}}}{2}=-m_2\tan\frac{\theta_1^{\mathrm{L}}}{2},$$
$$\theta_4^{\mathrm{L}}=\pi-\theta_1^{\mathrm{L}},\ \tan\frac{\theta_5^{\mathrm{L}}}{2}=\frac{(m_1m_2-1)\tan\frac{\theta_1^{\mathrm{L}}}{2}}{m_1+m_2\tan^2\frac{\theta_1^{\mathrm{L}}}{2}};\tag{5}$$

and

$$\tan\frac{\theta_2^{\mathrm{R}}}{2}=-\frac{m_3}{\tan\frac{\theta_1^{\mathrm{R}}}{2}},\ \tan\frac{\theta_3^{\mathrm{R}}}{2}=-m_4\tan\frac{\theta_1^{\mathrm{R}}}{2},$$
$$\theta_4^{\mathrm{R}}=\pi-\theta_1^{\mathrm{R}},\ \tan\frac{\theta_5^{\mathrm{R}}}{2}=\frac{(m_3m_4-1)\tan\frac{\theta_1^{\mathrm{R}}}{2}}{m_3+m_4\tan^2\frac{\theta_1^{\mathrm{R}}}{2}}.\tag{6}$$

respectively. Here, $m_i\ (i=1, 2, 3 \text{ and } 4)$ are used to simplify the following relationship.

$$m_1=\frac{\sin\frac{\beta+\alpha}{2}}{\sin\frac{\beta-\alpha}{2}},\ m_2=\frac{\sin\frac{\beta+\gamma}{2}}{\sin\frac{\beta-\gamma}{2}},\ m_3=\frac{\sin\frac{\delta+\alpha}{2}}{\sin\frac{\delta-\alpha}{2}}\ \text{and}\ m_4=\frac{\sin\frac{\delta+\gamma}{2}}{\sin\frac{\delta-\gamma}{2}}.\tag{7}$$

The relationship between revolute variables of the resultant 6*R* linkage and linkages L and R are

$$\theta_1=\theta_2^{\mathrm{L}},\ \theta_2=\theta_3^{\mathrm{L}},\ \theta_3=\pi-\theta_1^{\mathrm{L}}+\theta_1^{\mathrm{R}},$$
$$\theta_4=2\pi-\theta_3^{\mathrm{R}},\ \theta_5=2\pi-\theta_2^{\mathrm{R}},\ \theta_6=\theta_1^{\mathrm{L}}-\theta_1^{\mathrm{R}}-\pi.\tag{8}$$

The compatibility relationship between linkages L and R is

$$\theta_5^{\mathrm{L}}=\theta_5^{\mathrm{R}}.\tag{9}$$

Therefore, we can substitute Eqs. (5) and (6) into Eqs. (8) and (9) to derive the closure equations of the Form I linkage as

$$\tan\frac{\theta_2}{2}=\frac{m_1m_2}{\tan\frac{\theta_1}{2}},\ \theta_3=\pi+2\tan^{-1}\left(\frac{m_1}{\tan\frac{\theta_1}{2}}\right)+2\tan^{-1}P_{\theta_1},$$
$$\tan\frac{\theta_4}{2}=m_4P_{\theta_1},\ \tan\frac{\theta_5}{2}=\frac{m_3}{P_{\theta_1}},\ \theta_6=-\theta_3.\tag{10}$$

where

$$P_{\theta_1}=\frac{m_1m_2+\tan^2\frac{\theta_1}{2}}{2m_4\left(m_1m_2-1\right)\tan\frac{\theta_1}{2}}\left[\left(1-m_3m_4\right)\pm\sqrt{\left(1-m_3m_4\right)^2-\frac{4m_3m_4\left(m_1m_2-1\right)^2\tan^2\frac{\theta_1}{2}}{\left(m_1m_2+\tan^2\frac{\theta_1}{2}\right)^2}}\right].$$

Note that the above closure equations are in general form. When parameters of the geometry conditions changed, the value P_{θ_1} may apply to different domains to meet the condition of linkage closure. And the input-output curves of the Form I linkage are plotted in Fig. (6).

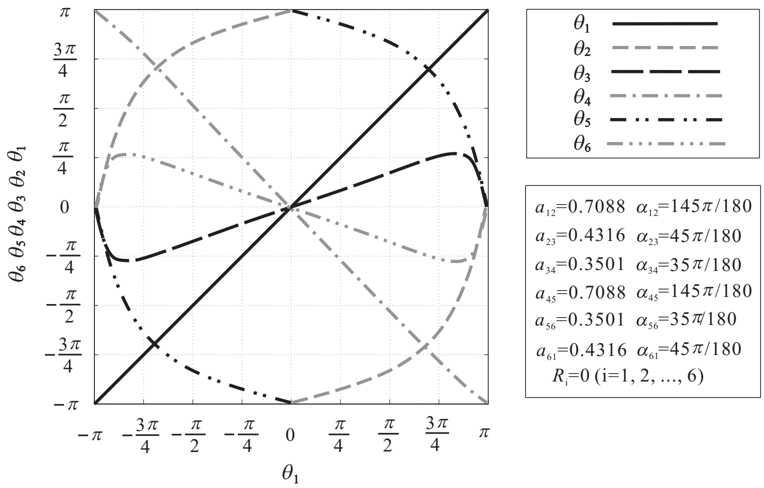

Figure 6: The input-output curves of Form I original double-Goldberg 6*R* linkage.

Similarly, we can achieve the Form I linkage by using linkages L2 and R1 as the construction bases, as shown in Fig. (7).

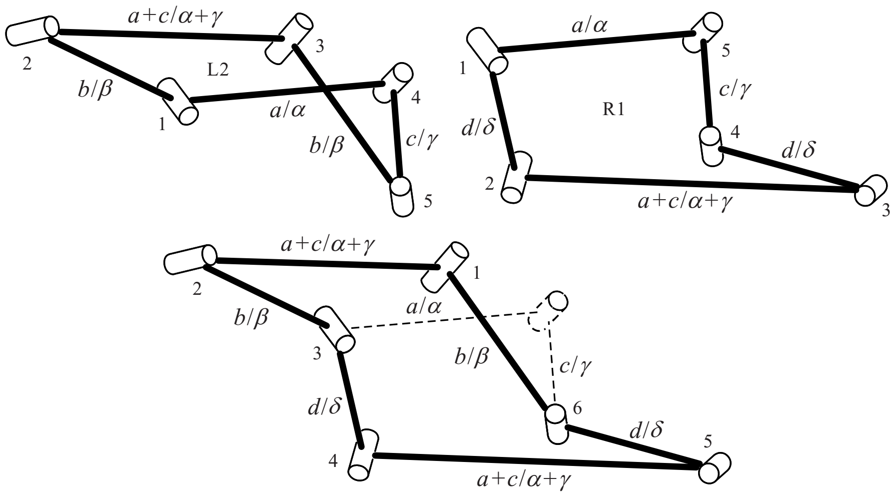

Figure 7: Construction of Form I original double-Goldberg 6*R* linkage by linkages L2 and R1.

3.2 Form II of the Original Double-Goldberg 6*R* Linkage

When using linkages L1 and R1 as the construction bases, through the same construction method as Form I linkage, a different form of the linkage, namely Form II linkage, will be obtained. As shown in Fig. (8), we can find that the geometry conditions of the Form II linkage are the same as the Form I linkage in Eq. (4). Even though they are derived from the same construction method, the Form II linkage has a different spatial layout as compared to the Form I linkage. Note that similar to the Form I linkage, when the parameters are changed, the value P_{θ_1} may apply to different domains to meet the condition of linkage closure.

$$\tan\frac{\theta_2}{2}=\frac{m_1m_2}{\tan\frac{\theta_1}{2}},\ \theta_3=\pi+2\tan^{-1}\left(\frac{m_1}{\tan\frac{\theta_1}{2}}\right)+2\tan^{-1}P_{\theta_1},$$
$$\tan\frac{\theta_4}{2}=m_4P_{\theta_1},\ \tan\frac{\theta_5}{2}=\frac{m_3}{P_{\theta_1}},\ \theta_6=-\theta_3, \tag{11}$$

where

$$P_{\theta_1}=\frac{m_1m_2+\tan^2\frac{\theta_1}{2}}{2m_4(m_1m_2-1)\tan\frac{\theta_1}{2}}\left[(1-m_3m_4)\mp\sqrt{(1-m_3m_4)^2-\frac{4m_3m_4(m_1m_2-1)^2\tan^2\frac{\theta_1}{2}}{\left(m_1m_2+\tan^2\frac{\theta_1}{2}\right)^2}}\right].$$

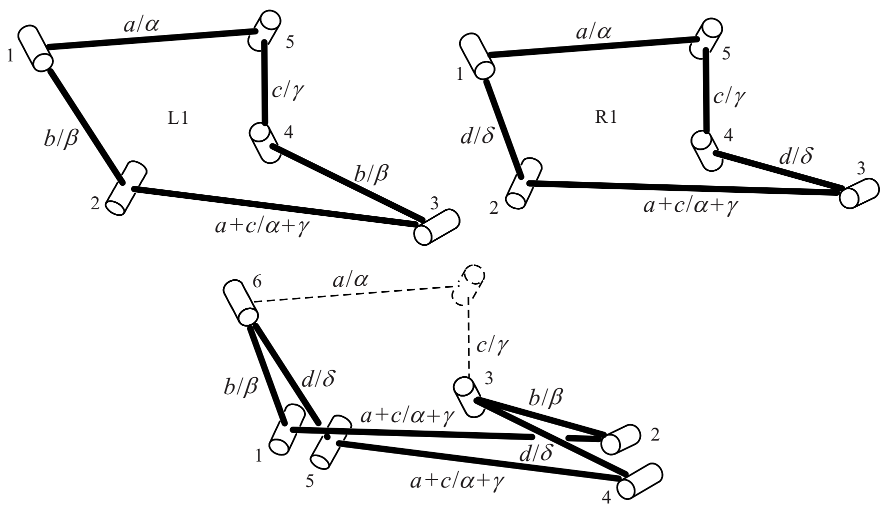

Figure 8: Construction of Form II original double-Goldberg 6*R* linkage by linkages L1 and R1.

The input-output curves of the Form II linkage are plotted in Fig. (9).

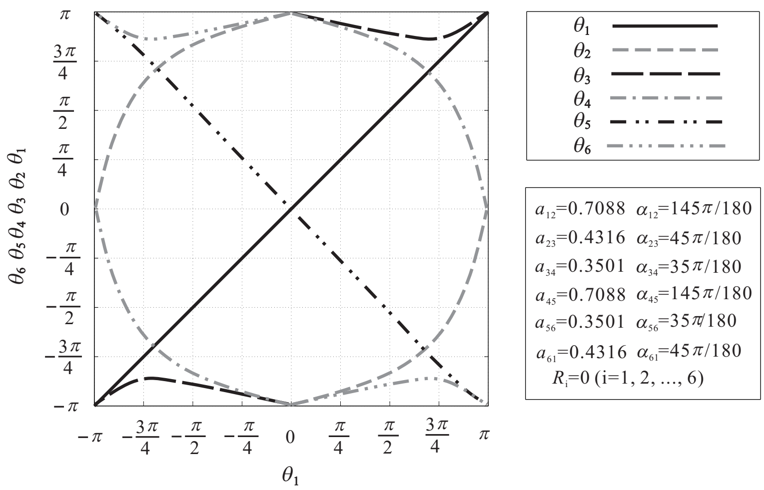

Figure 9: The input-output curves of Form II original double-Goldberg 6*R* linkage.

Alternatively, we can build the same linkage by using linkages L2 and R2 as the construction bases, as shown in Fig. (10).

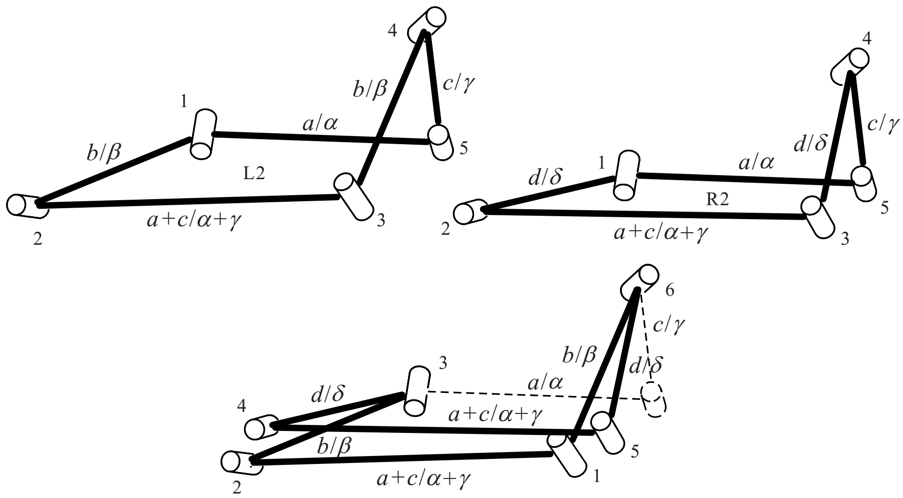

Figure 10: Construction of Form II original double-Goldberg 6*R* linkage by linkages L2 and R2.

4 BIFURCATION ANALYSIS OF THE ORIGINAL DOUBLE-GOLDBERG 6*R* LINKAGE

In order to take a further investigation into the kinematics of the original double-Goldberg 6*R* linkage, the method of Singular Value Decomposition (SVD) is used to examine the bifurcation behavior of the linkage. The SVD method is to solve the linkage's Jacobian matrix with a predictor and corrector step. Six singular values of the linkage's Jacobian matrix are monitored. When the sixth singular value remains zero, it indicates that the linkage has only one degree of freedom. When the fifth singular value falls to zero at some points, it indicates that the instantaneous mobility is increased in at these points. These points are the bifurcation points where the linkage might bifurcate into other kinematic paths. Plotted in Figs. (11) and (12) are the SVD results of the Forms I and II linkages. Bifurcation points are found in both figures when $\theta_1 = 0$ and $\theta_1 = \pm\pi$.

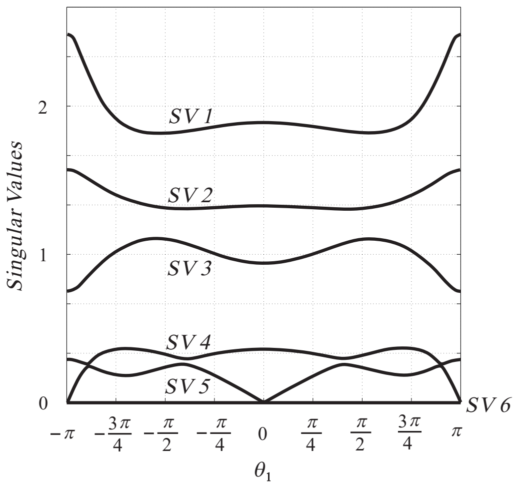

Figure 11: The SVD results of Form I original double-Goldberg 6*R* linkage.

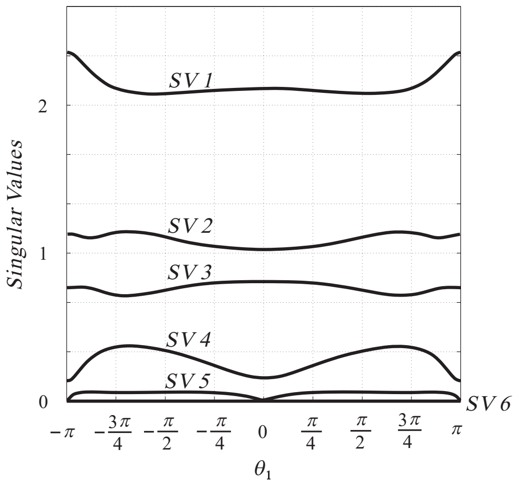

Figure 12: The SVD results of Form II original double-Goldberg 6*R* linkage.

4.1 Form III of the Original Double-Goldberg 6*R* Linkage

It is found that at the point $\theta_1 = 0$ in Fig. (11), the Form I linkage can bifurcate to another linkage form, namely the Form III linkage, as shown in Fig. (13). At the point $\theta_1 = \pm\pi$ in Fig. (12), the Form II linkage can bifurcate to the same linkage form.

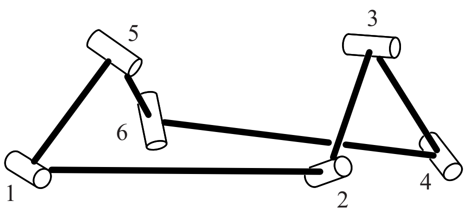

Figure 13: The Form III original double-Goldberg 6*R* linkage.

Different from Forms I and II linkages, the Form III linkage could not be decomposed into the combination between two Goldberg 5*R* linkages. By using the SVD method, we can plot the input-output curves of the linkage numerically, as shown in Fig. (14).

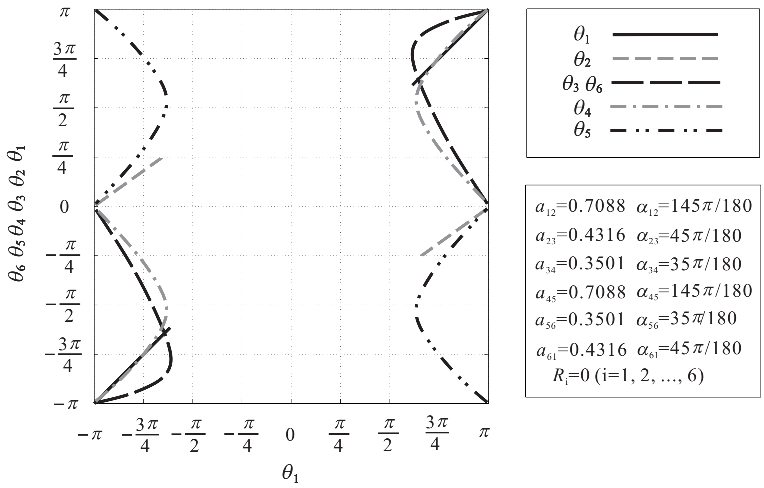

Figure 14: The input-output curves of Form III original double-Goldberg 6*R* linkage.

4.2 Form IV of the Original Double-Goldberg 6*R* Linkage

At the point $\theta_1 = \pm\pi$ in Fig. (11), the Form I linkage can bifurcate into the linkage form shown in Fig. (15), namely the Form IV linkage. At the point $\theta_1 = 0$ in Fig. (12), the Form II linkage can bifurcate to the same linkage form as well.

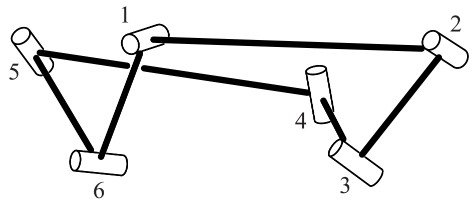

Figure 15: The Form IV original double-Goldberg 6*R* linkage.

The same as Form III linkage, the Form IV linkage cannot be decomposed into the combination between two Goldberg 5*R* linkages. By using the SVD method, we can plot the input-output curves of the linkage numerically, as shown in Fig. (16).

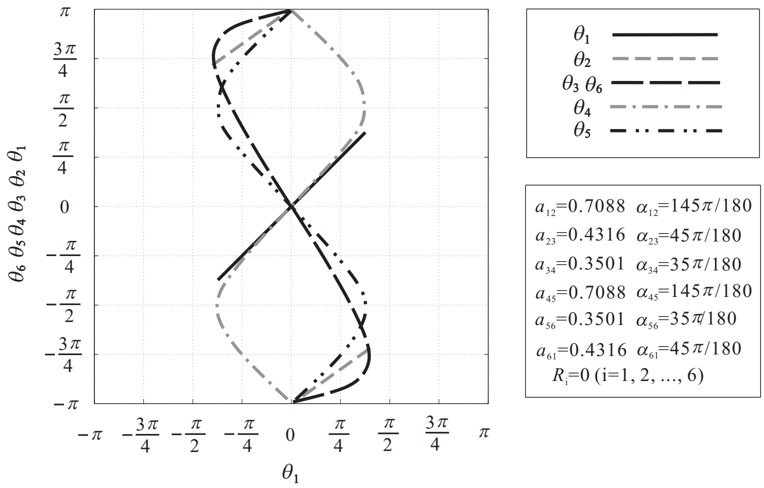

Figure 16: The input-output curves of Form IV original double-Goldberg 6*R* linkage.

Further investigation shows that the Forms III and IV linkages are in fact the same linkage in different numbering sequences. However, we still treat them as different linkage forms for the ease of differentiation. The relationship between the revolute variables of Forms III and IV linkages are

$$\theta_1^{\text{Form III}} = \theta_2^{\text{Form IV}},\ \theta_2^{\text{Form III}} = \theta_1^{\text{Form IV}},\ \theta_3^{\text{Form III}} = \theta_6^{\text{Form IV}},$$
$$\theta_4^{\text{Form III}} = \theta_5^{\text{Form IV}},\ \theta_5^{\text{Form III}} = \theta_4^{\text{Form IV}},\ \theta_6^{\text{Form III}} = \theta_3^{\text{Form IV}}. \tag{12}$$

In summary, we take the relationship between θ_5 and θ_1 as an example to demonstrate the transformation among these four forms of the original double-Goldberg 6*R* linkage. As shown in Fig. (17), (a)-(c) are the motion sequences of the Form I linkage; (d) is the bifurcation position $\mathrm{B_I}$ between the Forms I and III linkages; (e)-(g) are the motion sequences of Form III linkage; (h) is the bifurcation position $\mathrm{B'_{II}}$ between the Forms III and II linkages; (i)-(k) are the motion sequences of Form II linkage; (l) is the bifurcation point $\mathrm{B_{II}}$ between the Forms II and IV linkages; (m)-(o) are the motion sequences of the Form IV linkage; (p) is the bifurcation position $\mathrm{B'_I}$ between the Forms IV and I linkages.

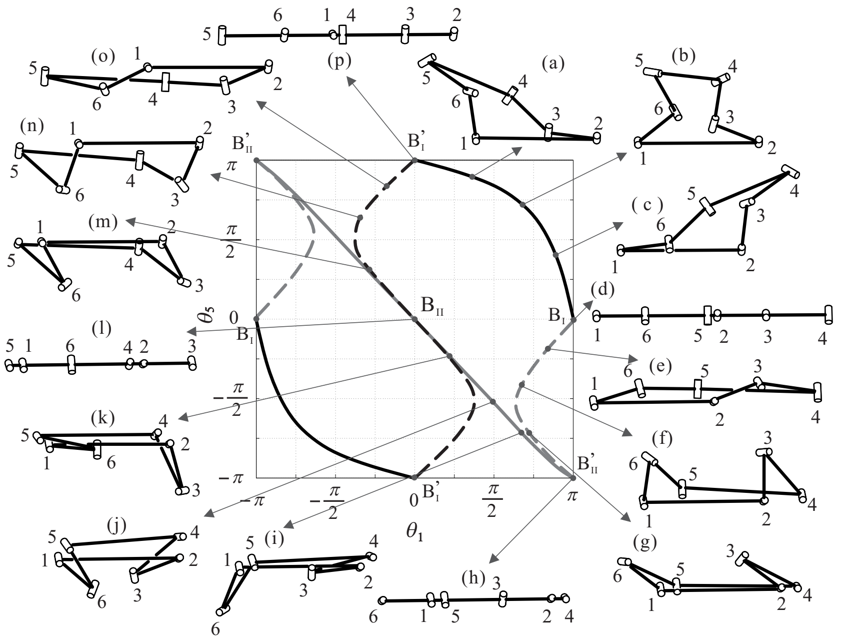

Figure 17: Transformation among different forms of the original double-Goldberg 6R linkage.

5 CONCLUSION

In this paper, the original double-Goldberg 6*R* linkage and its bifurcation behavior are studied at length. Because of the quadratic property of the revolute variable on the "roof-links" of the Goldberg 5*R* linkage, two different forms of the original double-Goldberg 6*R* linkages are achieved by merging two original Goldberg 5*R* linkages on the common "roof-links" and then removing this connection. By using the Singular Value Decomposition method, bifurcation points are located on both of the constructive forms of the linkage and two other forms of the linkage are found. The Forms I and II, which are derived from constructive method, are two different linkage forms; while the Forms III and IV, which are derived from numerical investigation, in fact share the same linkage layout. Moreover, Forms I and II cannot transform into each other directly, but they can only transform into each other by transform into either Forms III or IV first. Full transformation among these four forms of the original double-Goldberg 6*R* linkage makes it an interesting morphing structure to be studied.

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REFERENCES

[1] G. T. Bennett. A new mechanism. *Engineering*, vol. 76, 777–778, 1903.

[2] G. T. Bennett. The skew isogram mechanism. *Proceedings of the London Mathematical Society*, vol. 2, 151, 1914.

[3] R. Bricard, Leçons de cinématique, *Tome II Cinématique Appliquée*, pp. 7-12, 1927.

[4] M. Goldberg, New five-bar and six-bar linkages in three dimensions, *Transactions of the ASME*, vol. 65, 649–663, 1943.

[5] K. J. Waldron, Hybrid overconstrained linkages, *Journal of Mechanisms*, vol. 3, no. 2, 73-78, 1968.

[6] K. Wohlhart, Merging two general Goldberg 5*R* linkages to obtain a new 6*R* space mechanism, *Mechanism and Machine Theory*, vol. 26, no. 7, 659-668, 1991.

[7] Y. Chen, and Z. You, Spatial 6*R* linkages based on the combination of two Goldberg 5*R* linkages, *Mechanism and Machine Theory*, vol. 42, no. 11, 1484-1498, 2007.

[8] C. Y. Song, and Y. Chen, A spatial 6*R* linkage derived from subtractive Goldberg 5*R* linkages, *Mechanism and Machine Theory*, vol. 46, no. 8, 1097-1106, 2011.

[9] J. E. Baker, The Bennett, Goldberg and Myard linkages--in perspective, *Mechanism and Machine Theory*, vol. 14, no. 4, 239-253, 1979.

[10] J. E. Baker, A comparative survey of the Bennett-based, 6-revolute kinematic loops, *Mechanism and Machine Theory*, vol. 28, no. 1, 83-96, 1993.